Homework 4

due May 23

Problem 1. Recall that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has the basis E, F, H with relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

Let $\lambda \in \mathbb{C}$. The Verma module $\Delta(\lambda)$ is an infinite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ with basis

$$v_0 = v, \quad v_1 = Fv, \quad v_2 = F^2v, \quad v_3 = F^3v, \ldots$$

such that $Ev = 0$ and $Hv = \lambda v$. In particular, $Fv_k = v_{k+1}$.

(1) Find Ev_k and Hv_k for all $k \geq 0$.
(2) Suppose that λ is not a nonnegative integer. Prove that $\Delta(\lambda)$ is irreducible.
(3) Suppose that $\lambda = m$ is a nonnegative integer. Prove that $\Delta(m)$ has a unique proper submodule R_m, and the quotient $L(m) = \Delta(m)/R_m$ is finite-dimensional. Find the dimension of $L(m)$.
(4) Prove that $L(0)$ is trivial and $L(1)$ is isomorphic to the vector representation $\mathbb{C}^2$.
(5) Prove that $L(2)$ is isomorphic to the adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$.
(6) Suppose that $V = \bigoplus_m L(m)^{\oplus k_m}$. Let a_i denote the dimension of the H-eigenspace in V with eigenvalue i. Prove that $a_i = a_{-i}$, $a_i \geq a_{i+2}$ for $i \geq 0$ and $a_i \geq a_{i-2}$ for $i \leq 0$.

In the next problems, $L(m)$ is the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $m + 1$.

Problem 2. Use characters to prove $L(1) \otimes L(n) \cong L(n + 1) \oplus L(n - 1)$.

Problem 3. Decompose $L(a) \otimes L(b)$ into irreducible representations for all $a, b \in \mathbb{Z}_{\geq 0}$.

Problem 4. Decompose $L(1) \otimes L(1) \otimes L(1)$ into irreducible representations.

Problem 5. Prove that for all n the tensor product $\underbrace{L(1) \otimes \cdots \otimes L(1)}_{n}$ contains a unique copy of $L(n)$ and all other irreducible summands are of the form $L(j)$ with $j < n$.