Littlewood-Richardson rule / Clebsch Gordon coeff.

HW4

sl_2

$$L(a) \otimes L(b) = L(a+b) \oplus L(a+b-2) \oplus \cdots \oplus L(|a-b|)$$

irreducible repr.

Combinatorial description via crystal bases

introduced by Kashiwara around 1990's (Abel prize)
quantum groups $U_q(\mathfrak{g})$ as $q \to 0$

$L(1)$: $B(1)$ $\boxed{1} \xrightarrow{f} \boxed{2}$

$L(2)$: $B(2)$ $\boxed{1|1} \xrightarrow{f} \boxed{1|2} \xrightarrow{f} \boxed{2|2}$

$\vdots$

$L(m)$: $B(m)$ $\underbrace{\boxed{1|\cdots|1}}_{m} \xrightarrow{f} \underbrace{\boxed{1|\cdots|1|2}}_{m} \xrightarrow{f} \cdots \xrightarrow{f} \boxed{2|\cdots|2}$

Tensor products

$$v \otimes w \in B(a) \otimes B(b)$$

Assign (to 2,) to 1. Successively match brackets $\overset{2}{(}\ \overset{1}{)}$.

f: changes rightmost unbracketed 1 to 2

e: changes leftmost unbracketed 2 to 1

Example

Decomposition of $B(2)\otimes B(3)$:

$B(5)$

$$\boxed{1|1}\otimes\boxed{1|1|1} \longrightarrow \boxed{1|1}\otimes\boxed{1|1|2} \longrightarrow \cdots \longrightarrow \boxed{2|2}\otimes\boxed{2|2|2}$$

$B(3)$

$$\boxed{1|2}\otimes\boxed{1|1|1} \longrightarrow \boxed{1|2}\otimes\boxed{1|1|2} \longrightarrow \boxed{1|2}\otimes\boxed{1|2|2} \longrightarrow \boxed{2|2}\otimes\boxed{1|2|2}$$

$B(1)$

$$\boxed{2|2}\otimes\boxed{1|1|1} \longrightarrow \boxed{2|2}\otimes\boxed{1|1|2}$$

$$\Longrightarrow B(2)\otimes B(3) \cong B(5)\oplus B(3)\oplus B(1)$$

Definition

$v\otimes w \in B(a)\otimes B(b)$ is *highest weight* if $e(v\otimes w)=0$.

Highest weight elements in $B(a)\otimes B(b)$ are elements such that read right to left there are weakly more 1's than 2's.
(This implies that all 2's are bracketed).

Decomposition of $L(1)^{\otimes k}$:

Elements in $L(1)^{\otimes k}$ are words of length k in letters 1 and 2.

Highest weight elements are reverse lattice words in $\{1,2\}$ (from right to left weakly more 1's than 2's).

Via Robinson-Schensted-Knuth these are in bijection with standard Young tableaux of two row shape:

Ex 211212211 is h.w.

$$\begin{array}{|c|c|c|c|c|} \hline 3 & 4 & 6 & 9 \\ \cline{1-4} 1 & 2 & 5 & 7 & 8 \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{array}{c} 87654321 \\ 21121211 \end{array}$$

Letters with 1 go into bottom row of tableau, letters with 2 go into top row.

Decomposition:

$$L(1)^{\otimes k} \cong \bigoplus_{\substack{0 \le j \le k \\ j \equiv k \bmod 2}} c_j \, L(j)$$

$$c_j = \# \, SYT\left(\left(\frac{k+j}{2}, \frac{k-j}{2}\right)\right)$$

given by hook length formula

$$f^\lambda = \# SYT(\lambda) = \frac{n!}{\prod_{\substack{(i,j) \\ \text{cell in } \lambda}} h_\lambda(i,j)}$$

hook lengths

$$\begin{array}{|c|c|c|c|} \cline{1-1} 1 \\ \cline{1-1} 2 \\ \cline{1-2} 4 & 1 \\ \hline 7 & 4 & 2 & 1 \\ \hline \end{array}$$