Homework 2
due January 28, 2004

Question 1. Show that the complex derivative has the following properties (for problems (3)-(7) we assume that $f$ and $g$ are analytic on an appropriate open set):

1. If $f'(z_0)$ exists, then $f$ is continuous at $z_0$.

2. \[ \frac{d}{dz} z^n = nz^{n-1} \]

3. \[ \frac{d}{dz} (f + g) = \frac{df}{dz} + \frac{dg}{dz} \]

4. \[ \frac{d}{dz} (cf) = c \frac{df}{dz} \quad \text{for } c \in \mathbb{C} \]

5. \[ \frac{d}{dz} (fg) = \frac{df}{dz}g + f \frac{dg}{dz} \]

6. \[ \frac{d}{dz} \left( \frac{f}{g} \right) = \frac{\frac{df}{dz}g - f \frac{dg}{dz}}{g^2} \quad \text{for } g(z) \neq 0 \]

7. \[ \frac{d}{dz} f(g(z)) = f'(g(z))g'(z) \]

Question 2. Compute $df/dz$ when it exists for

(i) $f(z) = 1/z$
(ii) $f(z) = x^2 + iy^2$
(iii) $f(z) = z \text{Im}(z)$
(iv) $f(z) = z$

Question 3. Study

\[ f(z) = \begin{cases} \frac{xy(y-ix)}{x^2+y^2} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases} \]

Show that even though $|f(z) - f(0)|/z \to 0$ as $z \to 0$ for any straight line through the origin, $f(z)$ is nevertheless not complex differentiable at $z = 0$.

Question 4. Use the Cauchy–Riemann relations to show that an analytic function that takes only real values in some neighborhood, must be constant there.
**Question 5.** Define the symbols $\partial f / \partial z$ and $\partial f / \partial \overline{z}$ by

\[
\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{1}{i} \frac{\partial f}{\partial y} \right),
\]
\[
\frac{\partial f}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - \frac{1}{i} \frac{\partial f}{\partial y} \right).
\]

(i) Show that the Cauchy–Riemann equations are equivalent to $\partial f / \partial \overline{z} = 0$.
(ii) An antianalytic function is defined by the condition $\partial f / \partial z = 0$. Derive the Cauchy–Riemann relations for antianalytic functions.
(iii) Show that if $f$ is analytic, then $f' = \partial f / \partial z$.
(iv) If $f(z) = z$, show that $\partial f / \partial z = 1$ and $\partial f / \partial \overline{z} = 0$.
(v) If $f(z) = \overline{z}$, show that $\partial f / \partial z = 0$ and $\partial f / \partial \overline{z} = 1$.
(vi) Show that the symbols $\partial / \partial z$ and $\partial / \partial \overline{z}$ obey the sum, product, and scalar multiple rules for derivatives.
(vii) Show that the expression

\[
\sum_{n=0}^{N} \sum_{m=0}^{M} a_{nm} z^n \overline{z}^m
\]

with scalar $a_{nm} \in \mathbb{C}$ is an analytic function if and only if $a_{nm} = 0$ whenever $m \neq 0$.

**Question 6.** Let us study the Cauchy–Riemann relations in polar coordinates. In what follows, assume that $f(z)$ is analytic and that $z = r \exp(i \theta)$.

(i) Using that $df / dz$ can be computed for any direction of approach $dz \to 0$, show that

\[
\frac{df}{dz} = e^{-i \theta} \frac{\partial f}{\partial r}, \quad \frac{df}{dz} = \frac{1}{iz} \frac{\partial f}{\partial \theta}.
\]

(ii) Now, calling $f = u + iv$, deduce that

\[
\partial_r u = \frac{1}{r} \partial_r v, \quad \frac{1}{r} \partial_\theta u = -\partial_r v.
\]

(iii) These relations are just Cauchy–Riemann in polar coordinates. Check that they yield Laplace’s equation $\Delta u = 0 = \Delta v$ where $\Delta = \partial_r^2 + r^{-1} \partial_r + r^{-2} \partial_\theta^2$.
(iv) Verify that the function $u(r, \theta) = r^2 \cos(2 \theta)$ is harmonic. Use the polar form of Cauchy–Riemann to find the conjugate harmonic function $v(r, \theta)$. Express your final answer $f = u + iv$ in terms of $z$. 

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