MAT 246 University of California Winter 2005

Homework

Problem 1. Show that the dominance partial order on partitions of n satisfies

$$\lambda \trianglelefteq \mu \quad \Longleftrightarrow \quad \lambda' \trianglerighteq \mu',$$

where the prime denotes the transpose partition.

Problem 2. For $1 \leq i < j \leq n$, define the raising operator R_{ij} on $\mathbb{Z}^n$ by

$$R_{ij}(\nu_1, \ldots, \nu_n) = (\nu_1, \ldots, \nu_i + 1, \ldots, \nu_j - 1, \ldots, \nu_n).$$

(1) Show that the dominance order $\trianglelefteq$ is the transitive closure of the relation on partitions $\lambda \to \mu$ if $\mu = R_{ij}\lambda$ for some $i < j$.
(2) Show that μ covers λ if and only if $\mu = R_{ij}\lambda$, where i, j satisfy the following condition: either $j = i + 1$ or $\lambda_i = \lambda_j$ (or both).
(3) Find the smallest n such that the dominance order on partitions of n is not a total ordering, and draw its Hasse diagram.

Problem 3. Let $f \in \Lambda^n$, and for any $g \in \Lambda^n$ define $g_k \in \Lambda^{nk}$ by

$$g_k(x_1, x_2, \ldots) = g(x_1^k, x_2^k, \ldots).$$

Show that

$$\omega f_k = (-1)^{n(k-1)}(\omega f)_k.$$

Problem 4. The symmetric functions $f_\lambda = \omega m_\lambda$ are sometimes called the "forgotten" symmetric functions. Show that the matrix of coefficients of the forgotten symmetric functions f_λ expressed in terms of monomial symmetric functions m_λ is the transpose of the matrix of the elementary functions e_λ expressed in terms of the complete homogeneous symmetric functions h_λ.

Problem 5. Using the symmetry of the RSK algorithm, show the following:

(1) A permutation π is an involution if and only if $P(\pi) = Q(\pi)$, where $(P(\pi), Q(\pi))$ correspond to π under the RSK algorithm.
(2) The number of involutions of S_n is $\sum_{\lambda \vdash n} f^\lambda$.
(3) The number of fixed points in an involution π is the number of columns of odd length in $P(\pi)$.
(4) There is a bijection $M \longleftrightarrow T$ between symmetric $\mathbb{N}$-matrices of finite support and semistandard Young tableaux such that the trace of M is the number of columns of odd length of T.

(5) The following equations hold

$$\sum_{\lambda} s_\lambda = \prod_i \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j}$$
$$\sum_{\lambda' \text{ even}} s_\lambda = \prod_{i<j} \frac{1}{1-x_i x_j},$$

where λ' even means that every part in λ' is even.

Problem 6. Let ∂p_k be the operator on symmetric functions given by partial differentiation with respect to p_k, under the identification of symmetric functions with polynomials $f \in \mathbb{Q}[p_1, p_2, \ldots]$. Show that ∂p_k is adjoint with respect to the scalar product to the operator of multiplication by p_k/k.