Homework Set 1: Exercises on Complex Numbers

Directions: You are assigned the Calculational Problems 1(a, b, c), 2(b), 3(a, b), 4(b, c), 5(a, b), and the Proof-Writing Problems 8 and 11. Please submit your solutions to the Calculational and Proof-Writing Problems separately at the beginning of lecture on Friday January 12, 2007. The two sets will be graded by different persons.

1. Express the following complex numbers in the form $x + yi$ for $x, y \in \mathbb{R}$:

   (a) $(2 + 3i) + (4 + i)$
   (b) $(2 + 3i)^2(4 + i)$
   (c) $\frac{2 + 3i}{4 + i}$
   (d) $\frac{1}{i} + \frac{3}{1 + i}$
   (e) $(-i)^{-1}$

2. Compute the real and imaginary parts of the following expressions, where $z$ is the complex number $x + yi$ and $x, y \in \mathbb{R}$:

   (a) $\frac{1}{z^2}$
   (b) $\frac{1}{3z + 2}$
   (c) $\frac{z + 1}{2z - 5}$
   (d) $z^3$

3. Solve the following equations for $z$ a complex number:

   (a) $z^5 - 2 = 0$
   (b) $z^4 + i = 0$
   (c) $z^6 + 8 = 0$
(d) $z^3 - 4i = 0$

4. Calculate the

(a) complex conjugate of the fraction $(3 + 8i)^4/(1 + i)^{10}$.
(b) complex conjugate of the fraction $(8 - 2i)^{10}/(4 + 6i)^5$.
(c) complex modulus of the fraction $i(2 + 3i)(5 - 2i)/(-2 - i)$.
(d) complex modulus of the fraction $(2 - 3i)^2/(8 + 6i)^2$.

5. Compute the real and imaginary parts:

(a) $e^{2+i}$
(b) $\sin(1 + i)$
(c) $e^{3-i}$
(d) $\cos(2 + 3i)$

6. Compute the real and imaginary part of $e^{iz}$ for $z \in \mathbb{C}$.

7. Let $a \in \mathbb{R}$ and $z, w \in \mathbb{C}$. Prove that

(a) $\text{Re}(az) = a\text{Re}(z)$ and $\text{Im}(az) = a\text{Im}(z)$.
(b) $\text{Re}(z + w) = \text{Re}(z) + \text{Re}(w)$ and $\text{Im}(z + w) = \text{Im}(z) + \text{Im}(w)$.

8. Let $z \in \mathbb{C}$. Prove that $\text{Im}(z) = 0$ if and only if $\text{Re}(z) = z$.

9. Let $p$ be a polynomial with real coefficients and $z \in \mathbb{C}$.
   Show that $p(z) = 0$ if and only $p(\overline{z}) = 0$.

10. Let $z, w \in \mathbb{C}$. Prove the parallelogram law $|z - w|^2 + |z + w|^2 = 2(|z|^2 + |w|^2)$. 
11. Let \( z, w \in \mathbb{C} \) with \( \overline{z}w \neq 1 \) such that either \( |z| = 1 \) or \( |w| = 1 \). Prove that
\[
\left| \frac{z - w}{1 - \overline{z}w} \right| = 1.
\]

12. For an angle \( \theta \in [0, 2\pi) \), find the linear map \( f_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \) which describes the rotation by the angle \( \theta \) in the counterclockwise direction.

\textit{Hint:} For a given angle \( \theta \) find \( a, b, c, d \) such that \( f_\theta(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2) \).