Homework Set 7: More Exercises on Eigenvalues

Directions: Please work on all of the following exercises. Submit your solutions to Problems 3(d) and 5(b) as your Calculational Problems and Problems 1 and 2 as your Proof-Writing Problems at the **beginning** of lecture on February 23, 2007.

As usual, we are using $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

1. Let $a, b, c, d \in \mathbb{F}$ and consider the system of equations given by

$$\begin{aligned} ax_1 + bx_2 &= 0 \\ cx_1 + dx_2 &= 0. \end{aligned}$$

Note that $x_1 = x_2 = 0$ is a solution for any choice of a, b, c, and d. Prove that this system of equations has a non-trivial solution if and only if $ad - bc = 0$.

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}^{2\times 2}$, and recall that we can define a linear operator $T \in \mathcal{L}(\mathbb{F}^2)$ on $\mathbb{F}^2$ by setting $T(v) = Av$ for each $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{F}^2$.

Show that the eigenvalues for T are exactly the $\lambda \in \mathbb{F}$ for which $p(\lambda) = 0$, where $p(z) = (a - z)(d - z) - bc$.

Hint: Write the eigenvalue equation $Av = \lambda v$ as $(A - \lambda I)v = 0$ and use Problem 1.

3. Find eigenvalues and associated eigenvectors for the linear operators on $\mathbb{F}^2$ defined by the following 2×2 matrices:

(a) $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, (b) $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$, (c) $\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$, (e) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, (f) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hint: Use Problem 2.

4. For each matrix A below, find eigenvalues for the induced linear operator T on $\mathbb{F}^n$ without performing any calculations. Then describe the eigenvectors $v \in \mathbb{F}^n$ associated to each eigenvalue λ by looking at solutions to the matrix equation $(A - \lambda I)v = 0$, where I denotes the identity map on $\mathbb{F}^n$.

(a) $\begin{bmatrix} -1 & 6 \\ 0 & 5 \end{bmatrix}$, (b) $\begin{bmatrix} -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

5. For each matrix A below, describe the invariant subspaces for the induced linear operator T on $\mathbb{F}^2$ that maps each $v \in \mathbb{F}^2$ to $T(v) = Av$.

(a) $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$