LECTURE 1: COXETER GROUPS AND SCHUBERT CALCULUS

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Class website: http://www.math.ucdavis.edu/ anne/WQ2009/280.html

Recommended references:

- *Combinatorics of Coxeter Groups*, by Bjorner and Brenti;
- *Reflection Groups and Coxeter Groups*, by Humphreys;
- *Young Tableaux* by Fulton;
- *Symmetric Functions, Schubert Polynomials and Degeneracy Loci* by Manivel;
- *Notes on Schubert Polynomials* by Macdonald.

1. About Schubert Polynomials

Schubert polynomials were first introduced in 1982 by Lascoux and Schutzenberger. They are of great interest in mathematics, as they relate to combinatorics, representation theory and geometry. For example, they form a natural basis of the cohomology ring $H^*(G/B)$. They are also related to flag varieties and Grassmannians, etc.

2. The Symmetric Group

The symmetric group S_n is of primary importance in the study of Coxeter groups and Schubert polynomials. We define S_n as follows:

Definition 2.1. *Let S_n be the group generated by s_i, for $1 \leq i < n$, with relations:*

- $s_i^2 = 1$ *for all* $1 \leq i < n$*;*
- $s_i s_j = s_j s_i$ *if* $|i - j| \geq 2$*; and*
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ *for all* $1 \leq i < n$*.*

Alternatively, we can think of S_n as permuting the numbers $\{1, 2, \ldots, n\}$. We can represent a permutation using 1-line notation, say $\omega = [\omega_1, \omega_2, \ldots, \omega_n]$ where $\omega_i = \omega(i)$. For example, the permutation of $\{1, 2, 3\}$ that switches 1 and 2 and leaves 3 fixed is $\omega = [2, 1, 3]$. We can then view the elements s_i as transpositions that either switch the numbers in positions i and $i + 1$, or switch the locations of i and $i + 1$, depending on whether s_i acts on the right or the left.

Given an element ω of S_n, we can express ω as a minimal product of transpositions s_i. We call such an expression a *reduced expression*, which is not necessarily unique. We let $R(\omega)$ be the set of all reduced expressions of ω. If w is a reduced expression of ω, we let $\ell(w)$ = number of transpositions in w. By the following lemma, $\ell(\omega) = \ell(w)$ is well defined.

Lemma 2.2. *Given $w, v \in R(\omega)$, $\ell(w) = \ell(v)$.*

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One last thing we must note about the symmetric group is the existence of a unique longest element. In 1-line notation, this element is $[n, n-1, \ldots, 1]$, and it has length $\frac{(n-1)n}{2}$. We denote this element by ω_0.

3. Divided Difference Operators

Definition 3.1. *Let $K[X] := \mathbb{Z}[x_1, x_2, \ldots, x_n]$ be the polynomial ring over the integers in n variables.*

If $\omega \in S_n$, then S_n acts on $K[X]$ by $\omega(x_i) = x_{\omega(i)}$ for $i = 1, 2, ..., n$.

Definition 3.2. *We define the* divided difference operator*, $\partial_i : K[X] \to K[X]$, by*

$$\partial_i f(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n) - s_i f(x_1, \ldots, x_n)}{x_i - x_{i+1}}$$

for $1 \leq i < n$.

Given this definition, one can check the following relations:

(1) $\partial_i^2 = 0$
(2) $\partial_i \partial_j = \partial_j \partial_i$ for $|i-j| \geq 2$
(3) $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$

These three relations are checked explicitly below:

(1) Let $f \in K[X]$. Then

$$\begin{aligned}
\partial_i^2(f) &= \partial_i\left(\frac{f(x_1, \ldots, x_n) - s_i f(x_1, \ldots, x_n)}{x_i - x_{i+1}}\right) \\
&= \frac{\frac{f(x_1, \ldots, x_n) - s_i f(x_1, \ldots, x_n)}{x_i - x_{i+1}} - s_i\left(\frac{f(x_1, \ldots, x_n) - s_i f(x_1, \ldots, x_n)}{x_i - x_{i+1}}\right)}{x_i - x_{i+1}} \\
&= \frac{1}{(x_i - x_{i+1})^2}(f(x_1, \ldots, x_n) - s_i f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n) + s_i f(x_1, \ldots, x_n)) \\
&= 0
\end{aligned}$$

(2) Let $f \in K[X]$. Then

$$\begin{aligned}
\partial_i \partial_j f &= \frac{\frac{f(x_1, \ldots, x_n) - s_j f(x_1, \ldots, x_n)}{x_j - x_{j+1}} - s_i\left(\frac{f(x_1, \ldots, x_n) - s_j f(x_1, \ldots, x_n)}{x_j - x_{j+1}}\right)}{x_i - x_{i+1}} \\
&= \frac{1}{(x_i - x_{i+1})(x_j - x_{j+1})}[f(x_1, \ldots, x_n) - s_j(f(x_1, \ldots, x_n)) - s_i(f(x_1, \ldots, x_n)) + s_i s_j(f(x_1, \ldots, x_n))] \\
&= \partial_j \partial_i f
\end{aligned}$$

(3) Similar to above – simply expand using the definition, and apply the relation

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

Given the above three relations for divided difference operators, we can define the divided difference operator corresponding to a general element of the symmetric group:

Definition 3.3. *Given $\omega \in S_n$ and $w = s_{i_1} s_{i_2} \cdots s_{i_k} \in R(\omega)$, we let*

$$\partial_\omega = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k}$$

By the above relations, ∂_ω is well-defined and does not depend on the choice of reduced word. The algebra generated by ∂_i for $1 \leq i < n$ is known as the *nil-Hecke algebra*. Note that if we were to try to use a non-reduced word in the definition of ∂_ω, we would get 0 because $\partial_i^2 = 0$.

We can then define Schubert polynomials:

Definition 3.4. *For every $\omega \in S_n$, we define the* Schubert polynomial σ_ω *by:*

$$\sigma_\omega = \partial_{\omega^{-1}\omega_0}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 x_n^0)$$

where ω_0 is the unique longest element of S_n.

This is a straightforward definition; however, it is not ideal from a combinatorial standpoint since it involves applying a large number of divided difference operators. Billey, Jockusch and Stanley derived a more combinatorial formula (based on work by Fomin and Stanley) for Schubert polynomials that is presented below.

4. Combinatorial Definition of Schubert Polynomials

In the following, we identify a reduced word with the indices of that reduced word. For example, if $w = w_1 w_2 w_3 w_4 = s_3 s_1 s_2 s_1$ is a reduced expression for an element $\omega \in S_n$, we identify w with the word 3121, so statements such as $1 \leq w_1$ make sense.

Definition 4.1. *Let $\underline{a} = a_1 \cdots a_p \in R(\omega)$. We say that a p-tuple $\underline{\alpha} = (\alpha_1, \ldots, \alpha_p)$ of positive integers is $\underline{a}$ − compatible if:*

- $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p$;
- $\alpha_j \leq a_j$ *for all* $1 \leq j \leq p$; *and*
- $\alpha_j < \alpha_{j+1}$ *if* $a_j < a_{j+1}$.

Let $C(\underline{a})$ denote the set of $\underline{a}$-compatible sequences.

Theorem 4.2 (Fomin, Stanley 1991; Billey, Jockusch, Stanley 1993)**.**

$$\sigma_w = \sum_{\underline{a} \in R(w)} \sum_{\alpha \in C(\underline{a})} x_{\alpha_1} \cdots x_{\alpha_p}$$

Proof of this theorem is withheld until later in the class.

Example: Let $\omega = [3, 1, 2, 5, 4]$. Then we have $R(\omega) = \{214, 241, 421\}$. Note that we are writing reduced words as acting from left to right. We have to list all $\underline{a}$-compatible sequences for each reduced word.

- $w = 214$: $0 < \alpha_2 \leq 1$ so $\alpha_2 = 1$. Since α_i are weakly increasing, $\alpha_1 = 1$ as well. Then α_3 can be 2,3, or 4 since we need $\alpha_2 < \alpha_3 \leq a_3$.
- $w = 241$: There are no $\underline{a}$-compatible sequences because α_3 must be 1, but we have an ascent $a_1 < a_2$, so we must have $0 < \alpha_1 < \alpha_2 \leq \alpha_3 = 1$.

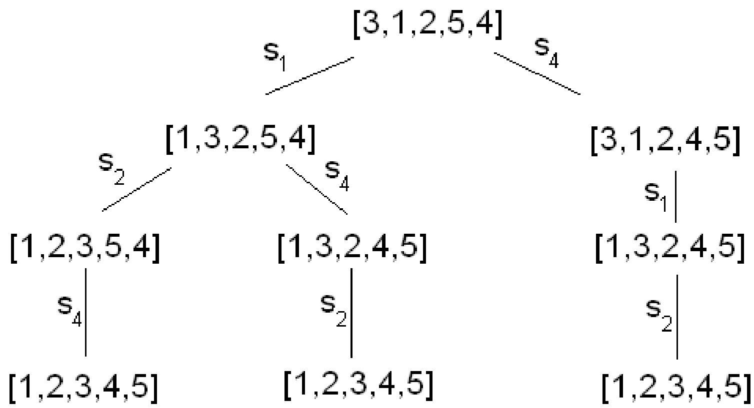

FIGURE 1. An algorithm to find the set of reduced words of [3,1,2,5,4].

- $w = 421$. $0 < \alpha_3 \leq 1$ so $\alpha_3 = 1$. This forces $\alpha_1 = \alpha_2 = 1$.

Therefore, $\alpha_w = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4$.

The set of reduced words $C(\omega)$ can be found by checking for descents in ω. If there is a descent at $\omega(i)$, multiply by s_i, and form a tree as in figure 1 to find the set of all inverses of reduced words of ω, from which it is trivial to find the set of reduced words of ω.

Note: For those interested in experimentation, SAGE (sagemath.org) can be very helpful. New functionality is being added daily, and it's free and open-source.

5. Coxeter Groups

Definition 5.1. *Let S be a set. A matrix $m : S \times S \to \{1, 2, \ldots, \infty\}$ is called a Coxeter matrix if:*

- *$m(s, s') = m(s', s)$ for all $s, s' \in S$*
- *$m(s, s') = 1 \iff s = s'$*

Definition 5.2. *A Coxeter graph is a graph with vertex set S and an undirected edge $\{s, s'\}$ if $m(s, s') \geq 3$. Additionally, we label the edge $\{s, s'\}$ by $m(s, s')$ if $m(s, s') \geq 4$.*

Example 5.3. *The following is a Coxeter matrix:*

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 2 \\ 3 & 4 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{pmatrix}$$

The Coxeter graph corresponding to this matrix is given in figure 2.

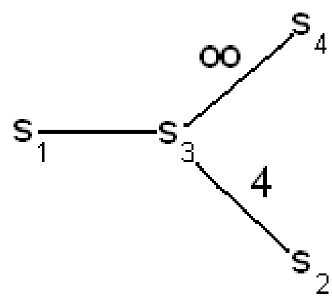

FIGURE 2. A Coxeter graph.

S. Billey, W. Jockusch, R. Stanley, *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2** (1993), no. 4, 345–374.

S. Fomin, R. Stanley, *Schubert polynomials and the nil-Coxeter algebra*, Adv. Math. **103** (1994), no. 2, 196–207.