LECTURE 10: WEAK BRUHAT ORDER

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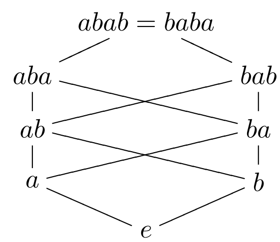
1. Lattices

Example 1.1. *Bruhat order in* B_2 *(Coxeter diagram:* $\bullet_a = \bullet_b$ *)*
$S = \{a, b\}$, $T = \{a, b, bab, aba\}$.
We can draw a graph showing the covering relations of Bruhat order on B_2*:*



Definition 1.2. *An element* z *in a poset is the* meet *(or greatest lower bound) of a subset* A *if*

(1) $z \le y, \quad \forall y \in A$
(2) $u \le y \quad \forall y \in A \Rightarrow u \le z$

We denote the meet of A *by* $\wedge A$*. If* $A = \{x, y\}$ *we denote the meet by* $x \wedge y$*.*

Note: If the meet exists, then it is unique.

Definition 1.3. *A poset* P *for which every* $\emptyset \ne A \subseteq P$ *has a meet is called a* meet-semilattice*.*

Definition 1.4. *Similarly, we can define the* join*, or least uppper bound, of a subset of a poset, and a* join-semilattice*. A* lattice *is a poset which is both a meet-semilattice and a join-semilattice.*

Note that the Bruhat graph in Example 1.1 above is not a lattice. However, when we can obtain a lattice if instead of Bruhat order we use *weak* Bruhat order.

2. Weak Bruhat Order

Weak Bruhat order is especially useful in studying the combinatorics of reduced words; for example, enumerating the number of reduced words of a given Coxeter group element. Intuitively, two elements are comparable in Bruhat order if one is a subword of the other. In weak Bruhat order, two words are comparable if one word is a prefix (or suffix) of the other. There are two weak orders, left and right weak Bruhat order, corresponding to if we are considering prefixes or suffixes.

Definition 2.1. *Let* (W, S) *be a Coxeter system, and let* $u, w \in W$*. Let* $\le_R$ *and* $\le_L$ *denote right and left (weak) Bruhat order, respectively. Then:*

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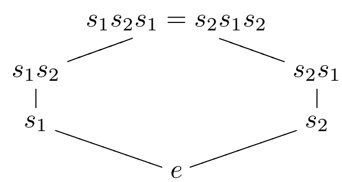
(1) $u \leq_R w$ *if* $w = us_1 \cdots s_k$*, where* $s_i \in S$*, s.t.* $\ell(us_1 \cdots s_i) = \ell(u) + i$*, for* $1 \leq i \leq k$*.*
(2) $u \leq_L w$ *if* $w = s_1 \cdots s_k u$*, where* $s_i \in S$*, s.t.* $\ell(s_1 \cdots s_i u) = \ell(u) + i$*, for* $1 \leq i \leq k$*.*

Remark 2.2. *Note that left and right weak orders are distinct, but they are isomorphic by the map* $w \to w^{-1}$*.*

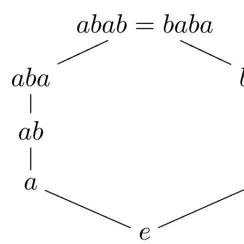
Weak Bruhat order called "weak" because $u \leq_R w \Rightarrow u \leq w$, and $u \leq_L w \Rightarrow u \leq w$.

Example 2.3. *We can draw the covering relations for weak Bruhat order:*

(1) *Let* $W = S_3$*.*



(2) *Let* $W = B_2$*.*



Note that in the examples above, we do get lattices.

In the case of S_n, there is a simple test: for $x, y \in S_n$, $x \leq_R y \Leftrightarrow y$ can be obtained from x by a sequence of adjacent transpositions that increase the inversion number at each step.

Example 2.4. *Let* $263154 \in S_6$ *be given in 1-line notation. We can multiply by* s_4 *on the right (acting on positions) to get* 263514*, which increases the inversion number. We could then multiply by* s_1*, then* s_5*, then* s_2 *to get the sequence:*

$$x = 263154 \to^{s_4} 263514 \to^{s_1} 623514 \to^{s_5} 623541 \to^{s_2} 632541 = y$$

Therefore, $x \leq_R y$*.*

Proposition 2.5. *Properties of Weak Order*

(1) *There is a 1-1 correspondence between reduced words for* $w \in W$ *and maximal chains in* $[e, w]_R$*.*
(2) $u \leq_R w \Leftrightarrow \ell(u) + \ell(u^{-1}w) = \ell(w)$*.*
(3) *If* W *is finite, then* $w \leq w_0$ *for all* $w \in W$*.*
(4) *Prefix property:* $u \leq_R w \Leftrightarrow$ *there exist reduced expressions* $u = s_1 \cdots s_k$ *and* $w = s_1 \cdots s_k s_{k+1} \cdots s'_k$*.*
(5) *Chain property:* $u <_R w \Rightarrow$ *there is a chain* $u = u_0 <_R u_1 <_R \cdots <_R u_k = w$ *such that* $\ell(u_i) = \ell(u_{)}) + i$ *for* $0 \leq i \leq k$*.*
(6) *Let* $s \in D_L(u) \cap D_L(w)$*. Then* $u \leq_R w \Leftrightarrow su \leq_R sw$*.*

Proposition 2.6. *Let $u, w \in W$. Then $u \leq_R w \Leftrightarrow T_L(u) \subseteq T_L(w)$, where $T_L(u) = \{t \in T \mid \ell(tu) \leq \ell(u)\}$.*

Proof. ($\Rightarrow$) Let $u = s_1 \cdots s_k$, $w = s_1 \cdots s_k \cdots s_q$ be reduced words. Then $T_L(u) = \{s_1 s_2 \cdots s_i \cdots s_2 s_1 \mid 1 \leq i \leq k\} \subseteq \{s_1 s_2 \cdots s_i \cdots s_2 s_1 \mid 1 \leq i \leq q\} = T_L(w)$.

($\Leftarrow$) Suppose $u = s_1 \cdots s_k$ is reduced. Let $t_i = s_1 s_2 \cdots s_i \cdots s_2 s_1$ for $1 \leq i \leq k$. Assume $T_L(u) = \{t_1, \cdots, t_k\} \subseteq T_L(w)$. We claim there is a reduced expression $w = s_1 \cdots s_i s_1' \cdots s_{q-i}'$, for $0 \leq i \leq k$. For $i = 0$, this is trivially true since this just means there exists a reduced word for w. Now suppose the claim is true for some i, $0 \leq i < k$. By assumption, $t_{i+1} \in T_L(w)$. We know that $t_j \neq t_{i+1}$ for $j \leq i$ by a lemma from a previous lecture (using that $s_1 \cdots s_{i+1}$ is reduced). Then since we can write $w = s_1 \cdots s_i s_1' \cdots s_{q-i}'$, we can write $t_{i+1} = s_1 \cdots s_i s_1' \cdots s_m' \cdots s_1' s_i \cdots s_1$ for some $1 \leq m \leq q - i$. Then

$$\begin{aligned} w = t_{i+1}^2 w &= (s_1 \cdots s_{i+1} \cdots s_1)(s_1 \cdots s_i s_1' \cdots \hat{s_m'} \cdots s_{q-i}') \\ &= s_1 \cdots s_{i+1} s_1' \cdots \hat{s_m'} \cdots s_{q-i}'. \end{aligned}$$

Then $u \leq_R w$ is equivalent to the claim for $i = k$ by the Prefix Property. $\square$

Corollary 2.7. *$w \to T_L(w)$ provides an order and rank-preserving embedding $W \hookrightarrow$lattice of finite subsets of T.*

Proposition 2.8. *If W is finite,*

(1) *$w \to w_0 w$ and $w \to w w_0$ are anti-automorphisms of weak order and*
(2) *$w \to w_0 w w_0$ is an automorphism of weak order.*

Proof. We will prove (2), as (1) is similar.

For all $s \in S$, $s w_0 = w_0 s'$ for some $s' \in S$, since $w_0 S w_0 = S$. Suppose $w \leq_R ws$. Then $w_0 w s w_0 = w_0 w w_0 s' \leq_R w_0 w w_0$ since $\ell(w_0 w s w_0) = \ell(ws) = \ell(w) + 1 = \ell(w_0 w w_0) + 1 > \ell(w_0 w w_0)$. $\square$