LECTURE 11: PARABOLIC SUBGROUPS

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1. Results for left and right order

Proposition 1.1. *Translation principle If $u \leq_R w$, then $[u, w]_R \cong \left[e, u^{-1}w\right]_R$*

Proof. We want to show that $x \mapsto ux$ is a poset isomorphism. $[e, u^{-1}w]_R \to [u, w]_R$. Note

$$\ell(w) = \ell(u) + \ell(u^{-1}w) \tag{1.1}$$
$$\leq \ell(u) + \ell(x) + \ell(x^{-1}u^{-1}w) \tag{1.2}$$
$$\geq \ell(ux) + \ell(x^{-1}u^{-1}w) \tag{1.3}$$
$$\geq \ell(w). \tag{1.4}$$

$$\begin{aligned} x \leq_R u^{-1}w &\Leftrightarrow equality\ at(1.2) \\ &\Leftrightarrow equality\ at(1.3)and(1.4) \\ &\Leftrightarrow u \leq_R ux \leq_R w. \end{aligned}$$

Hence $X \in [e, u^{-1}w]_R \Leftrightarrow ux \in [u, w]_R$ and also $\ell(ux) = \ell(u) + \ell(x)$. □

Corollary 1.2. *Let $u \leq_R w$, $m = \ell(u^{-1}w) \Rightarrow \sharp\{v \in [u, w]_R | \ell(v) = \ell(u) + k\} \leq \binom{m}{k}$*

Proof. Follows from the Boolean embedding.

$$\begin{aligned} w &\mapsto T_L(w) \\ u \leq_R w &\Leftrightarrow T_L(u) \leq T_L(w) \end{aligned}$$

□

Theorem 1.3. *Weak order on W is a complete meet - semilattice*

Proof. Bjorner- Brenti, Thm. 3.2.1. □

2. Parabolic subgroups

$T \subseteq S$
W_J is a subgroup of W generated by J and it is called a parabolic subgroup.

Proposition 2.1. *(1) (W_J, J) is a Coxeter group.*
(2) $\ell_J(w) = \ell(w)$ for every $w \in W_J$.
(3) $W_I \bigcap W_J = W_{I \bigcap J}$
(4) $< W_I \bigcup W_J >= W_{I \bigcup J}$
(5) $W_I = W_J \Rightarrow I = J$

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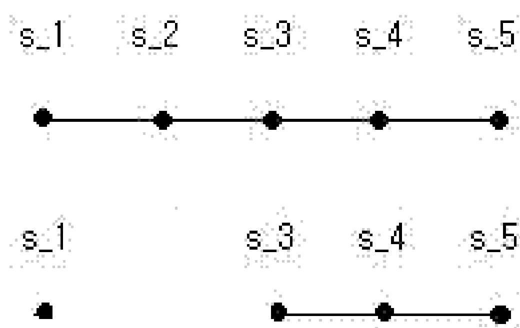
Proof. $w \in W_J, w = s_1 \cdots s_k$ for some $s_i \in J$. By the Deletion Property we may assume that this word is reduced $\Rightarrow w \in W_J \Rightarrow \ell_J(w) = \ell(w) \Rightarrow (2)$.
Since $\ell_J(w) = \ell(w)$ we can use the Exchange Property holds in W_J as a special case of the Exchange property in W
$\Rightarrow$ by the Characterization Theorem we have the (W_J, J) is a Coxeter System $\Rightarrow (1)$
$(3), (5)$by the Exchange property, (4) an easy exercise. □

Remark 2.2. Coxeter diagram of (W_J, J) is obtained by removing all modes $S \setminus J$

Example 2.3. S_6
$S = \{s_1, \cdots, s_5\}$, $J = S \setminus \{s_2\}$, $W_J = S_2 \times S_4$



In general $S_n, J = S \setminus \{s_k\}$ and $W_j \cong S_k \times S_{n-k}$

Definition 2.4. If W_J is finite $\Rightarrow$ it has a maximal element denoted by $w_0 J$, $w_0(\oslash) = e$, $w_0(S) = w_0$ if W is finite.

Definition 2.5.

$$D_I^J := \{w \in W | I \subseteq D_R(w) \subseteq J\}$$
$$W^J := D_{\oslash}^{S\setminus J} = \{w \in W, |ws > w \forall s \in J\}$$
$$D_I := D_I^I$$

Lemma 2.6. *$w \in W^J \Leftrightarrow$ no reduced expression for w ends with a letter in J.*

Proposition 2.7. *If $J \subseteq S$ then we have:*
(1) Every $w \in W$ has a unique factorization $w = w^J w_J$ such that $w^J \in W^J$ and $w_J \in W_J$
(2) $\ell(w) = \ell(w^J) + \ell(w_J)$.

Proof. Existence
We choose $s_1 \in J$ such that $ws_1 < w$ (if it exists).
We continue choosing $s_i \in J$ such that $ws_i \cdots s_i < ws_i \cdots s_{i-1}$ as long as it exists.
Process has to step after at most $\ell(w)$ steps. If it ends at step k then $w_k = ws_i \cdots s_k$ satisfies $w_k s > w_k \forall S \in J \Rightarrow w_k \in W^J$.
Let $v = s_k \cdots s_1 \in W_J \Rightarrow w = w_k v$ and by construction we have $\ell(w) = \ell(w_k) + k$

Uniqueness
We suppose $w = uv = xy$ with $u, x \in W^J, v, y \in W_J$
Let $u = s_1 s_2 \cdots s_k$ reduced, $s_i \in S$ and $vy^{-1} = s_1' \cdots s_q'$ (not necessarily reduced) with $s_i' \in J$
$\Rightarrow x = uvy^{-1} = s_1 \cdots s_k s_1' \cdots s_q'$
From this we can extract a reduced expression for x by deleting some elements. Therefore it cannot end in s_j' since $x \in W^J$
Therefore the reduced word for x has to be a subword of $s_1 \cdots s_k \Rightarrow x \leq u$. But by symmetry we can also deduce that $u \leq x$ so therefore $x = u \Rightarrow v = y$. □

3. Divided difference operators

Newton's divided difference operators. They act on polynomials in n variables : ∂_i
$(\partial_i f)(x_1, \cdots, x_n) = \frac{f(x_1,\ldots,x_n) - f(x_1,\ldots,x_{i+1},x_i,\ldots,x_n)}{x_i - x_{i+1}}$
or $\partial_i = (x_i - x_{i+1})^{-1}(1 - s_i)$

Remark 3.1. Space of symmetric polynomial in x_i and x_{i+1} are both kernel and the image of ∂_i

Lemma 3.2. *For every f, g (polynomials) $\partial_i(fg) = (\partial_i f)_g + (s_i f)(\partial_i g)$*

Proof. Exercise.

□