Definition 0.1. Let \( X = \{x_1, \cdots, x_n\} \), then \( s_i \in S_n \) acts on \( f \in \mathbb{Z}[X] \) by switching the \( x_i \) and \( x_{i+1} \). That is,
\[
s_i \cdot f(x_1, \cdots, x_{i}, x_{i+1}, \cdots, x_n) = f(x_1, \cdots, x_{i}, x_{i+1}, \cdots, x_n)
\]
For \( s_i \in S_n \), define \( \partial_i : \mathbb{Z}[X] \to \mathbb{Z}[X] \) by
\[
(\partial_i f)(x_1, \cdots, x_n) = \frac{f(x_1, \cdots, x_n) - s_i \cdot f(x_1, \cdots, x_n)}{x_i - x_{i+1}}
\]
In another word, \( \partial_i = (x_i - x_{i+1})^{-1}(1 - s_i) \).

1. Graph of reduced words

Given \( w \in (W, S) \), we can define a colored graph \( \Gamma(w) \), called the graph of reduced words of \( w \), as follow. The nodes in this graph are the set of all reduced words of \( w \). Let \( u, v \) be two nodes of this graph, (i.e., two reduced work of \( w \) then \( u, v \) are connected by an edge colored (labeled) by a defining relation of \( (W, S) \) if and only if \( u \) can be transformed to \( v \) (or vise versa) by one application of the given defining relation.

It is clear that the defining relations \( s_i^2 = e \) are never used in \( \Gamma(w) \), for all nodes in \( \Gamma(w) \) are reduced words.

Example 1.1. Let the Coxeter system be \( (S_5, \{s_1, s_2, s_3, s_4\}) \), and let \( w = [31542] \) in one-line notation, then one reduced word for \( w \) could be \( s_2s_3s_4s_3s_1 \), we will just code this word by its indices \( 23431 \). Let us show the connected component of \( \Gamma(w) \) that contains this word. (As we will see later this is the whole graph)

\[
\begin{align*}
23431 & \quad 23413 & \quad 24341 & \quad 42341 \\
23413 & \quad 23143 & \quad 24314 & \quad 42314 \\
& \quad 21343 & \quad 24134 & \quad 42134 \\
& \quad 21434
\end{align*}
\]

Proposition 1.2. Given \( w \in (W, S) \), \( \Gamma(w) \) is connected.

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Proof. We prove the statement for the case of \((S_n, \{s_1, \cdots, s_{n-1}\})\), the spirits are the same for all other types of Coxeter group.

Induct on \(\ell(w)\). For the case \(\ell(w) \leq 2\) the statement clearly holds.

Now assume \(\ell(w) \geq 3\), let \(a = a_1 \cdots a_l\) and \(b = b_1 \cdots b_l\) be two reduced words of \(w\). By E.P. \(b_1w\) has a reduced expression of the form \(a_1 \cdots \hat{a}_k \cdots a_l\), for some \(1 \leq k \leq l\). Thus \(b_1a_1 \cdots \hat{a}_k \cdots a_l\) is another reduced expression of \(w\).

First note that by induction, \(b_2 \cdots b_l\) and \(a_1 \cdots \hat{a}_k \cdots a_l\) are connected by a sequence of edges in \(\Gamma(b_1w)\), thus \(b_1b_2 \cdots b_l\) and \(b_1 \cdots \hat{a}_k \cdots a_l\) are connected by a sequence of edges in \(\Gamma(w)\).

If \(k < l\), then in \(\Gamma(wa_1)\), \(a_1 \cdots a_{l-1}\) and \(b_1 \cdots \hat{a}_k \cdots a_l\) are connected by a sequence of edges, thus in \(\Gamma(w)\), \(a_1 \cdots a_l\) and \(b_1 \cdots \hat{a}_k \cdots a_l\) are connected by a sequence of edges. Therefore \(a\) and \(b\) are connected in \(\Gamma(w)\).

If \(k = l\), then either \(a_1\) and \(b_1\) are consecutive or not. If they are not, then \(b_1a_1 \cdots a_{l-1}\) and \(a_1b_1a_2 \cdots a_{l-1}\) are connected by a single edge labeled by \(a_1b_1 = b_1a_1\). Now in \(\Gamma(w)\), \(a\) and \(a_1b_1a_2 \cdots a_{l-1}\) are connected by a sequence of edges "lifted" from \(\Gamma(a_1w)\), thus \(a\) and \(b\) are connected in \(\Gamma(w)\).

Finally, if \(k = l\) but \(a_1\) and \(b_1\) are consecutive, then by E.P. \(a_1b_1a_1 \cdots \hat{a}_j \cdots a_{l-1}\) for some \(1 \leq j \leq l - 1\) is another reduced word of \(w\). If \(j = 1, \cdots, l - 2\) then we just repeat about argument for the case \(k < l\) with \(b' = a\) and \(a' = b_1a_1 \cdots a_{l-1}\). Otherwise if \(j = l - 1\), in particular \(j > 1\), then \(a_1b_1a_1 \cdots \hat{a}_j \cdots a_{l-1}\) is connected to \(b_1a_1b_1 \cdots \hat{a}_j \cdots a_{l-2}\) by an edge labeled by \(a_1b_1a_1 = b_1a_1b_1\) in \(\Gamma(w)\). Now we note that \(b\) and \(a_1b_1a_1 \cdots \hat{a}_j \cdots a_{l-1}\) are connected in \(\Gamma(w)\) by lifting a path from \(\Gamma(b_1w)\), and \(a\) and \(a_1b_1a_1 \cdots \hat{a}_j \cdots a_{l-1}\) are connected in \(\Gamma(w)\) by lifting a path from \(\Gamma(a_1w)\), we are done. \(\square\)

2. Properties of divided difference operators

Lemma 2.1. If \(f, g \in \mathbb{Z}[X]\) then
\[
\partial_i(f \ast g) = (\partial_i f) \ast g + (s_i \cdot f) \ast (\partial_i g)
\]

Proof.
\[
\partial_i(f \ast g) = \frac{f \ast g - s_i \cdot (f \ast g)}{x_i - x_{i+1}}
\]
\[
= \frac{f \ast g - (s_i \cdot f) \ast g + (s_i \cdot f) \ast g - s_i \cdot (f \ast g)}{x_i - x_{i+1}}
\]
\[
= \frac{f \ast g - (s_i \cdot f) \ast g + (s_i \cdot f) \ast g - (s_i \cdot f) \ast (s_i \cdot g)}{x_i - x_{i+1}}
\]
\[
= (\partial_i f) \ast g + (s_i \cdot f) \ast (\partial_i g)
\]

\(\square\)

Theorem 2.2 (Nil-Coxeter relations).
\[
\partial_i \partial_j = \partial_j \partial_i \text{ for } |i - j| > 1
\]
\[
\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \text{ for } i = 1, \cdots, n - 1
\]
\[
\partial_i^2 = 0
\]

Proof. In lecture note 1, Steven and Alex gave a detailed proof. \(\square\)

Definition 2.3. If \(a_1 \cdots a_l\) is a reduced word of \(w \in S_n\), then define \(\partial_w = \partial_{a_1} \cdots \partial_{a_l}\).
Remark 2.4. Because of the fact that $\Gamma(w)$ is connected (where edges in $\Gamma$ correspond to only the first two nil-Coxeter relations), $\partial_w$ does not depend on the choice of reduced word, thus is well-defined.

On the other hand, if $a_1 \cdots a_l$ is not a reduced word, then $\partial_{a_1} \cdots \partial_{a_l} = 0$. To see that we let $1 \leq j < l$ be such that $u = a_1 \cdots a_j$ is a reduced word but $a_1 \cdots a_{j+1}$ is no longer reduced. Then there is another reduced expression of $u$ that is ended with $a_{j+1} b_1 \cdots a_{j+1}$. Now $\partial_{a_1} \cdots \partial_{a_j} \partial_{a_{j+1}} \partial_{a_l} = \partial_{a_1} \cdots \partial_{a_{j+1}} \partial_{a_{j+1}} \cdots \partial_{a_l} = 0$ by the third relation.

From above consideration, we can conclude

(*) \[ \partial_u \partial_v = \begin{cases} \partial_{uv} & \text{if } \ell(uv) = \ell(u) + \ell(v) \\ 0 & \text{otherwise} \end{cases} \]

Lemma 2.5. $s_i \circ \partial_w = \partial_w$ if and only if $\ell(iw) = \ell(w) - 1$

Proof. $s_i \circ \partial_w = \partial_w \Leftrightarrow \partial_i \partial_w = 0$ (by definition)

\[ \Leftrightarrow \ell(s_i w) = \ell(w) - 1 \ (\text{by } *) \]

\[ \square \]

Proposition 2.6. If $w_0$ is the longest element of $S_n$, then

\[ \partial_{w_0} = a_\delta^{-1} \sum_{w \in S_n} \epsilon(w) w \]

where

\[ a_\delta = \prod_{1 \leq i < j \leq n} (x_i - x_j) \]

is the Vandermonde determinant, and $\epsilon(w)$ is the sign of $w$.

Proof. By definition, for any $v \in S_n$, $\partial_v$ can be written of the form

\[ \partial_v = \sum_{w \in S_n} c_{v,w} w \]

In particular, we can write $\partial_w = \sum_{w \in S_n} c_{w,w} w$. By above lemma, we have $s_i \partial_{w_0} = \partial_{w_0}$ for any $i = 1, \cdots, n - 1$, thus $u \partial_{w_0} = \partial_{w_0}$ for any $u \in S_n$. This implies

\[ \sum_{w \in S_n} c_{w,w} w = \sum_{w \in S_n} (u, c_w)(uw) \]

Comparing the coefficient, we get $c_{uw} = u, c_w$. Thus if we know the coefficient $c_w$ for some $w$, then we can derive $c_w$ for all $w$. Indeed, we claim that we know

\[ c_{w_0} = a_\delta^{-1} \epsilon(w_0) \]

Assume above claim, we note $w = w w_0 w_0$, thus $c_w = w w_0 c_{w_0} = \epsilon(w) \epsilon(w_0) a_\delta^{-1} = \epsilon(w) a_\delta^{-1}$.

The only thing left to show is the claim. One reduced expression of $w_0$ is

\[ w_0 = (s_{n-1} \cdots s_1)(s_{n-1} \cdots s_2) \cdots (s_{n-1}) \]

So,

\[ \partial_{w_0} = (\partial_{n-1} \cdots \partial_1)(\partial_{n-1} \cdots \partial_2) \cdots (\partial_{n-1}) \]
We are interested in the coefficient of \( w_0 \) after "multiply out" the rhs. For \( n = 3 \), the least non-trivial case we see

\[
\partial_{w_0} = \partial_2 \partial_1 \partial_2 = \frac{1}{x_2 - x_3} (1 - s_2) \frac{1}{x_1 - x_2} (1 - s_1) \frac{1}{x_2 - x_3} (1 - s_2)
\]

clearly \( c_{w_0} = \frac{1}{x_2 - x_3} \frac{1}{x_1 - x_2} \frac{1}{x_2 - x_3} (-1)^3 \), claim shown.

The general case can be checked explicitly in a similar fashion. □