LECTURE 14: YANG-BAXTER EQUATION AND DOUBLE SCHUBERT POLYNOMIALS

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1. The Yang-Baxter Equation (Continued). Double Schubert Polynomials.

Last time we talked about the nil-Coxeter algebra, and we saw that the nil-Coxeter relations for $u_1, u_2, \dots u_{n-1}$ are given by

$$\begin{aligned} u_i u_j &= u_j u_i \quad \text{for } |i-j| > 1 \\ u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} \\ {u_i}^2 &= 0 \end{aligned}$$

We showed the following result for $h_i(x) = 1 + x u_i(x)$.

Lemma 1.1.

$$\begin{aligned} h_i(x)h_i(y) &= h_i(x+y) \\ h_i(x)h_j(y) &= h_j(y)h_i(x) \quad |i-j| > 1 \\ h_i(x)h_j(x+y)h_i(y) &= h_j(y)h_i(x+y)h_j(x) \quad |i-j| = 1 \end{aligned}$$

The last relation is called the Yang-Baxter equation.

We also learned that we can associate to a strand configuration $\mathcal{C}$ a polynomial

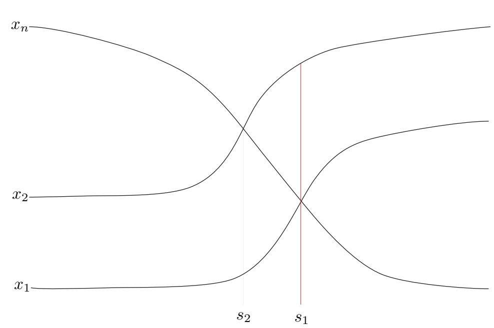

Figure 1. Strand representation

$\Phi(\mathcal{C})$ in $\mathcal{H}[x]$. In the above example $\Phi(\mathcal{C}) = h_{s_2}(x_3 - x_2)h_{s_1}(x_3 - x_1)$.

Next we consider a particular configuration, as shown in the following figure.

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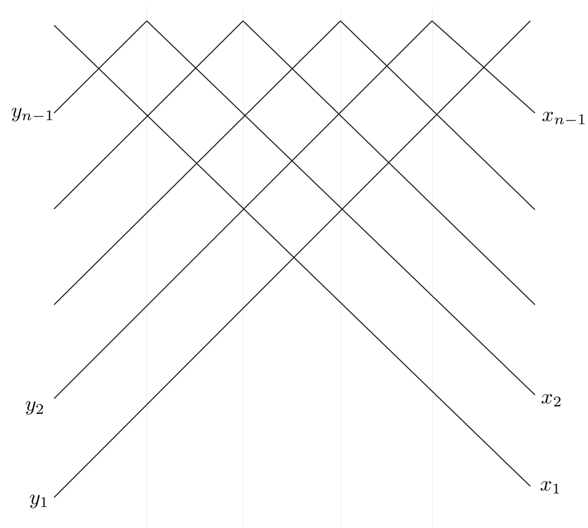

FIGURE 2. Particular configuration

$$\Phi(C_{\mathrm{sp}}) = \prod_{d=2-n}^{n-2} \prod_{\substack{i-j=d \\ i+j\leq n}} h_{i+j-1}(x_i - y_j).$$

Note that the order of the factors in the product is important! Deforming Fig. 2 we obtain Fig. 3 (using only braid and commutation relations which we showed last time do not change $\Phi(\mathcal{C}_{\mathrm{sp}})$), which simplifies the calculation of Φ.

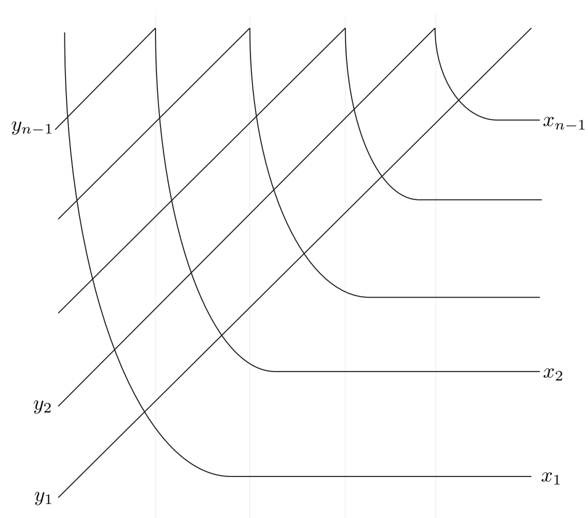

FIGURE 3. Simplified particular configuration

By this simple observation it is easy to see that we can rewrite Φ as:

$$\Phi(\mathcal{C}_{\text{sp}}) = \prod_{i=1}^{n-1} \prod_{j=n-i}^{1} h_{i+j-1}(x_i - y_j)$$

where recall $h_i(x) = 1 + xu_i$.

Theorem 1.2. *If one decomposes $\Phi(\mathcal{C}_{sp})$ in $\mathcal{H}[x, y]$ as*

$$\Phi(\mathcal{C}_{sp}) = \sum_{w \in S_n} \Phi_w(\mathcal{C}_{sp})w$$

then

$$\Phi_w(\mathcal{C}_{sp}) = \sigma_w(x, y).$$

Proof. Let us first look at

$$\Phi_{w_0}(\mathcal{C}_{\text{sp}}) = \prod_{i+j \le n} (x_i - y_j) = \Delta(x, y) = \sigma_{w_0}(x, y)$$

Recall $\partial_i \sigma_w = \sigma_{ws_i}$ if $\ell(ws_i) = \ell(w) - 1$. Hence it remains to show that the same recursion holds for the coefficient polynomials in $\Phi(\mathcal{C}_{\text{sp}})$. But we have that

$$\partial_i \Phi_{ws_i}(\mathcal{C}_{\text{sp}}) = \Phi_w(\mathcal{C}_{\text{sp}}) \quad \text{for } \ell(ws_i) = \ell(w) - 1$$

if and only if

$$\partial_i \Phi(\mathcal{C}_{\text{sp}}) = \Phi(\mathcal{C}_{\text{sp}})u_i.$$

Set

$$H_i(x) = h_{n-1}(x) \cdots h_{i+1}(x)h_i(x),$$

Then note that

$$\begin{aligned} H_i(x) &= H_{i+1}(x)h_i(x), \\ h_i(x)H_j(y) &= H_j(y)h_i(x) \text{ if } j > i + 1, \\ h_i(x)h_i(-x) &= 1. \end{aligned}$$

Lemma 1.3.
(a) $H_i(x)H_i(y) = H_i(y)H_i(x)$
(b) $H_i(x)H_{i+1}(y) - H_i(y)H_{i+1}(x) = (x - y)H_i(x)H_{i+1}(y)u_i$.

Proof.
(a) This follows be descending induction on i:

$$\begin{aligned} H_i(x)H_i(y) &= H_{i+1}h_i(x)H_{i+2}(y)h_{i+1}(y)h_i(y) \\ &= H_{i+1}(x)H_{i+2}(y)h_i(x)h_{i+1}(y)h_i(y - x)h_i(x) \\ &= H_{i+1}(x)H_{i+2}(y)h_{i+1}(y - x)h_i(y)h_{i+1}(x)h_i(x) \\ &\qquad \text{(this latter is theY-B eq.)} \\ &= H_{i+1}(x)H_{i+1}(y)h_{i+1}(-x)h_i(y)h_{i+1}(x)h_i(x) \\ &= H_{i+1}(y)H_{i+1}(x)h_{i+1}(-x)h_i(y)h_{i+1}(x)h_i(x) \\ &\qquad \text{(but } H_{i+2}(x) = H_{i+1}(x)h_{i+1}(-x)) \\ &= H_{i+1}(y)h_i(y)H_{i+2}(x)h_{i+1}(x)h_i(x) \\ &= H_i(y)H_i(x) \end{aligned}$$

(b)

$$
\begin{aligned}
H_i(x)H_{i+1}(y) - H_i(y)H_{i+1}(x) &= H_i(x)H_i(y)h_i(-y) - H_i(y)H_i(x)h_i(-x) \\
&\qquad (\text{and } h_i(-y) = 1 - yu_i, \ \ h_i(-x) = 1 - xu_i) \\
&= H_i(x)H_i(y)(-yu_i) + H_i(x)H_i(y)xu_i \\
&= (x-y)H_i(x)H_i(y)u_i \\
&= (x-y)H_i(x)H_{i+1}(y)(1+yu_i)u_i \\
&\qquad (\text{but } 1 + yu_i = 0 \ ) \\
&= (x-y)H_i(x)H_{i+1}(y)u_i
\end{aligned}
$$

□

Lemma 1.4.
(a) $h_i(x-y) = {H_{i+1}}^{-1}(x){H_i}^{-1}(y)H_i(x)H_{i+1}(y)$
(b) $h_{n-1}(x-y_{n-1})\cdots h_i(x-y_i) = {H_{n-1}}^{-1}(y_{n-1})\cdots {H_i}^{-1}(y_i)H_i(x)H_{i+1}(y_i)\cdots H_n(y_{n-1})$

Proof.
(a) Observe that the equality is equivalent to

$$H_i(y)H_{i+1}(x)h_i(x)h_i(-y) = H_i(x)H_{i+1}(y)$$

but this latter is equivalent to $H_i(y)H_i(x) = H_i(x)H_i(y)$, which corresponds precisely to part (a) of the previous lemma.
(b) This part can be proved by descending induction on i and the previous lemma. We leave the details to the reader. □

We now complete the proof of Theorem 1.2. Using Lemma 1.4 (b) we find that

$$
\begin{aligned}
\Phi(\mathcal{C}_{\text{sp}}) &= \prod_{i=1}^{n-1} \prod_{j=n-i}^{1} h_{i+j-1}(x_i - y_j) \\
&= \sigma^{-1}(y)\sigma(x)
\end{aligned}
$$

where $\sigma(x) = H_1(x_1)H_2(x_2)\cdots H_{n-1}(x_{n-1})$. Hence it remains to show that

$$\partial_i \sigma(x) = \sigma(x)u_i.$$

But we can see that

$$
\begin{aligned}
\partial_i \sigma(x) &= \frac{H_1(x_1)\cdots H_{n-1}(x_{n-1}) - H_1(x_1)\cdots H_i(x_{i+1})H_{i+1}(x_i)\cdots H_{n-1}(x_{n-1})}{(x_i - x_{i+1})} \\
&= H_1(x_1)\cdots H_{n-1}(x_{n-1})u_i \\
&= \sigma(x)u_i
\end{aligned}
$$

by Lemma 1.3 (b). □