LECTURE 14: YANG-BAXTER EQUATION AND DOUBLE SCHUBERT POLYNOMIALS

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1. THE YANG-BAXTER EQUATION (CONTINUED). DOUBLE SCHUBERT POLYNOMIALS.

Last time we talked about the nil-Coxeter algebra, and we saw that the nil-Coxeter relations for $u_1, u_2, \ldots, u_{n-1}$ are given by

\[
\begin{align*}
    u_i u_j &= u_j u_i & \text{for } |i - j| > 1 \\
    u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} \\
    u_i^2 &= 0
\end{align*}
\]

We showed the following result for $h_i(x) = 1 + xu_i(x)$.

**Lemma 1.1.**

\[
\begin{align*}
    h_i(x)h_i(y) &= h_i(x + y) \\
    h_i(x)h_j(y) &= h_j(y)h_i(x) & |i - j| > 1 \\
    h_i(x)h_j(x + y)h_i(y) &= h_j(y)h_i(x + y)h_j(x) & |i - j| = 1
\end{align*}
\]

The last relation is called the Yang-Baxter equation.

We also learned that we can associate to a strand configuration $\mathcal{C}$ a polynomial

\[
\Phi(\mathcal{C}) \in \mathcal{H}[x]. \text{ In the above example } \Phi(\mathcal{C}) = h_{s_2}(x_3 - x_2)h_{s_1}(x_3 - x_1).
\]

Next we consider a particular configuration, as shown in the following figure.

\[\text{Figure 1. Strand representation}\]

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\[ \Phi(C_{sp}) = \prod_{d=2}^{n-2} \prod_{\begin{array}{c} i-j=d \\ i+j \leq n \end{array}} h_{i+j-1}(x_i - y_j). \]

Note that the order of the factors in the product is important! Deforming Fig. 2 we obtain Fig. 3 (using only braid and commutation relations which we showed last time do not change \( \Phi(C_{sp}) \)), which simplifies the calculation of \( \Phi \).
By this simple observation it is easy to see that we can rewrite $\Phi$ as:

$$\Phi(C_{sp}) = \prod_{i=1}^{n} \prod_{j=n-i}^{1} h_{i+j-1}(x_i - y_j)$$

where recall $h_i(x) = 1 + xu_i$.

**Theorem 1.2.** If one decomposes $\Phi(C_{sp})$ in $H[x, y]$ as

$$\Phi(C_{sp}) = \sum_{w \in S_n} \Phi_w(C_{sp})w$$

then

$$\Phi_w(C_{sp}) = \sigma_w(x, y).$$

**Proof.** Let us first look at

$$\Phi_{w_0}(C_{sp}) = \prod_{i+j \leq n} (x_i - y_j) = \Delta(x, y) = \sigma_{w_0}(x, y)$$

Recall $\partial \sigma_w = \sigma_{ws_i}$ if $\ell(ws_i) = \ell(w) - 1$. Hence it remains to show that the same recursion holds for the coefficient polynomials in $\Phi(C_{sp})$. But we have that

$$\partial_i \Phi_{ws_i}(C_{sp}) = \Phi_w(C_{sp})$$

for $\ell(ws_i) = \ell(w) - 1$

if and only if

$$\partial_i \Phi(C_{sp}) = \Phi(C_{sp})u_i.$$  

Set

$$H_i(x) = h_{n-1}(x) \cdots h_{i+1}(x)h_i(x),$$

Then note that

$$H_i(x) = H_{i+1}(x)h_i(x),$$

$$h_i(x)H_j(y) = H_j(y)h_i(x) \text{ if } j > i + 1,$$

$$h_i(x)h_i(-x) = 1.$$

**Lemma 1.3.**

(a) $H_i(x)H_i(y) = H_i(y)H_i(x)$

(b) $H_i(x)H_{i+1}(y) - H_i(y)H_{i+1}(x) = (x - y)H_i(x)H_{i+1}(y)u_i.$

**Proof.**

(a) This follows be descending induction on $i$:

$$H_i(x)H_i(y) = H_{i+1}(x)H_{i+2}(y)h_{i+1}(y)h_i(y)$$

$$= H_{i+1}(x)H_{i+2}(y)h_i(x)h_{i+1}(y)h_i(y-x)h_i(x)$$

$$= H_{i+1}(x)H_{i+2}(y)h_{i+1}(y-x)h_i(y)h_{i+1}(x)h_i(x)$$

(this latter is theY-B eq.)

$$= H_{i+1}(x)H_{i+1}(y)h_{i+1}(-x)h_i(y)h_{i+1}(x)h_i(x)$$

$$= H_{i+1}(y)H_{i+1}(x)h_{i+1}(-x)h_i(y)h_{i+1}(x)h_i(x)$$

(but $H_{i+2}(x) = H_{i+1}(x)h_{i+1}(-x)$)

$$= H_{i+1}(y)h_i(y)H_{i+2}(x)h_{i+1}(x)h_i(x)$$

$$= H_i(y)H_i(x)$$

$$= H_i(x)H_i(y)$$
(b) \[ H_i(x)H_{i+1}(y) - H_i(y)H_{i+1}(x) = H_i(x)H_i(y)h_i(-y) - H_i(y)H_i(x)h_i(-x) \]

(\text{and } h_i(-y) = 1 - yu_i, \ h_i(-x) = 1 - xu_i)

\[ = H_i(x)H_i(y)(-yu_i) + H_i(x)H_i(y)xu_i \]

\[ = (x-y)H_i(x)H_i(y)u_i \]

\[ = (x-y)H_i(x)H_{i+1}(y)(1 + yu_i)u_i \]

(\text{but } 1 + yu_i = 0)

\[ = (x-y)H_i(x)H_{i+1}(y)u_i \]

\[ \square \]

Lemma 1.4.

(a) \( h_i(x-y) = H_{i+1}^{-1}(x)H_i^{-1}(y)H_i(x)H_{i+1}(y) \)

(b) \( h_{n-1}(x-y_{n-1}) \cdots h_1(x-y_1) = H_{n-1}^{-1}(y_{n-1}) \cdots H_i^{-1}(y_i)H_i(x)H_{i+1}(y_i) \cdots H_n(y_{n-1}) \)

Proof.

(a) Observe that the equality is equivalent to

\[ H_i(y)H_{i+1}(x)h_i(x)h_i(-y) = H_i(x)H_{i+1}(y) \]

but this latter is equivalent to \( H_i(y)H_i(x) = H_i(x)H_i(y) \), which corresponds precisely to part (a) of the previous lemma.

(b) This part can be proved by descending induction on \( i \) and the previous lemma.

We leave the details to the reader. \( \square \)

We now complete the proof of Theorem 1.2. Using Lemma 1.4 (b) we find that

\[ \Phi(C_{sp}) = \prod_{i=1}^{n-1} \prod_{j=n-i}^{1} h_{i+j-1}(x_i - y_j) \]

\[ = \sigma^{-1}(y)\sigma(x) \]

where \( \sigma(x) = H_1(x_1)H_2(x_2) \cdots H_{n-1}(x_{n-1}) \). Hence it remains to show that

\[ \partial_i\sigma(x) = \sigma(x)u_i. \]

But we can see that

\[ \partial_i\sigma(x) = \frac{H_1(x_1) \cdots H_{n-1}(x_{n-1}) - H_1(x_1) \cdots H_i(x_{i+1})H_{i+1}(x_i) \cdots H_{n-1}(x_{n-1})}{(x_i - x_{i+1})} \]

\[ = H_1(x_1) \cdots H_{n-1}(x_{n-1})u_i \]

\[ = \sigma(x)u_i \]

by Lemma 1.3 (b). \( \square \)