LECTURE 16: COMBINATORIAL FORMULA FOR SINGLE SCHUBERT POLYNOMIALS AND RC-GRAPHS

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1. Combinatorial formula for single Schubert polynomials

Theorem 1. Combinatorial Theorem:

\[ \sigma_w(x) = \sum_{a \in R(w)} \sum_{b \in C(a)} x_{b_1} \ldots x_{b_\ell} \]

where \( C(a) \) is the set of increasing \( a \)-compatible words, \( \ell \) is the length of \( w \), and

1. \( b_1 \leq b_2 \leq \cdots \leq b_\ell \)
2. \( b_i \leq a_i \)
3. \( b_i < b_{i+1} \) if \( a_i < a_{i+1} \)

Proof. We have

\[ \phi(C_{SP}) \big|_{y=0} = \prod_{i=1}^{n-1} \prod_{j=n-i}^1 h_{i+j-1}(x_i) = \sigma(x) \]

(Recall: \( h_i(x) = 1 + xu_i \), where the \( u_i \)'s satisfy the nilCoxeter algebra) We need to expand the product and look for the coefficient of \( w \); the \( b_i \)'s are indices of the \( x_i \)'s, and each \( h_{i+j-1} \) contributes \( u_{i+j-1} \).

We get part (2) from the fact that \( i \leq i + j - 1 \), and we get (3) because since the product \( \prod_{j=n-i}^1 \) is decreasing, we must have \( b_i < b_{i+1} \) if \( a_i < a_{i+1} \).

Example 2. Consider \( S_3 \). Then \( \sigma(x) = h_2(x_1)h_1(x_1)h_2(x_2) = (1 + x_1u_2)(1 + x_1u_1)(1 + x_2u_2) \). Note that \( (1 + x_1u_2)(1 + x_1u_1) \) from the term \( i = 1 \) in the inner product, which is decreasing, and \( (1 + x_2u_2) \) comes from \( i = 2 \).

Aim 1: We want to prove that the Schubert polynomials \( \sigma_w(x), w \in S_\infty \), form an integral basis for \( \mathbb{Z}[x_1, x_2, \ldots] \).

Aim 2: Monk’s Rule—expansion of \( \sigma_w \sigma_{s_i} \)

2. RC-graps


Definition 3. Let \( a = a_1a_2 \ldots a_p \in R(w) \) and \( \alpha = \alpha_1 \ldots \alpha_p \in C(a) \). The reduced-word compatible sequence graph or rc-graph for short is

\[ D(a, \alpha) = \{ (\alpha_k, a_k - \alpha_k + 1) \mid 1 \leq k \leq p \} \]

Set

\[ RC(w) = \{ D(a, \alpha) \mid a \in R(w), \alpha \in C(a) \} \]
Example 4. $a = 521345, \alpha = 111235$

The plus signs indicate positions in $D(a, \alpha)$; note that if $(i, j) \in D$, then $i + j \leq n$ if $w \in S_n$.

Algorithm to get $w \in S_n$ from graph:

Each line alternates between going up and going to the right unless it hits a plus sign, in which case it goes through. Follow the strand labelled $i$ from left to write to obtain $w(i)$.

In the example we have $w = [3, 1, 4, 6, 5, 2]$ (because $w(1) = 3, w(2) = 1$, etc.); $\ell(w) = 6$ since we have 6 crossings.

Note that strands do not cross more than once.

Remark 5. The transpose $D^t$ of an rc-graph $D \in \mathcal{RC}(w)$ is an rc-graph in $\mathcal{RC}(w^{-1})$.

Denote by $\rho: \mathcal{RC}(w) \to \mathcal{RC}(w^{-1})$ the bijection mapping $D \mapsto D^t$.

Notation: For $D \in \mathcal{RC}(w)$ let $x_D = \prod_{(i, j) \in D} x_i$.

Corollary 6.

$$\sigma_w(x) = \sum_{D(a, \alpha) \in \mathcal{RC}(w)} x^{D(a, \alpha)}$$
3. Moves on rc-graphs

Let $w \in S_\infty$ and $D \in \mathcal{RC}(w)$. A ladder move $L_{ij}$ is defined as:

A chute move $C_{ij}$ is defined as:

Remark 7. $\rho(L_{ij}(D)) = C_{ji}(\rho(D))$, i.e. $L_{ij}$ and $C_{ij}$ are dual to each other.

Lemma 8. Ladder and chute moves preserve the permutation associated to $D$:

$\text{perm}C_{ij}(D) = \text{perm}(D)$

$\text{perm}L_{ij}(D) = \text{perm}(D)$

Proof. We use a proof by picture. The strands in the region of a chute move look like this:

Following each strand one can easily check that each letter still gets mapped to the same position. □

Lemma 9. $D \in \mathcal{RC}(w)$ is the result of a chute move (or, equivalently, admits an inverse chute move) if and only if there exists $(i, j) \notin D$ such that $(i + 1, j) \in D$.

Remark 10. Geometrically, an inverse chute move cannot be applied if all +’s in each column are clumped at the top.
Proof. Suppose \((i, j) \notin D\) and \((i + 1, j) \in D\). Look right along row \(i + 1\) for the smallest \(k > j\) such that \((i + 1, k) \notin D\) (There must be such a \(k\) since \(D\) contains only finitely many \(+\)).

Claim: \((i, k) \notin D\)

Proof. Suppose this is not true, i.e. \((i, k) \in D\). Then we would have:

This is impossible because strands cannot cross twice. □

Let \(m\) be the position of the last dot before \(k\), that is \(m < k\) largest such that \((i, m) \notin D\). Therefore the \(+\) at \((i + 1, m)\) can be moved to \((i, k)\) by an inverse chute move. □