The goal of this section is to use rc-graphs to show that the set of Schubert polynomials form a basis for $\mathbb{Z}[x_1, x_2, \ldots]$. Previously we established a combinatorial formula for the Schubert polynomials, which was a sum over reduced words $a$ for an element $w \in S_n$ and $a$-compatible sequences, $\alpha$. We also defined the rc-graph $D$ associated to a pairing $(a, \alpha)$, defined the Chute and Ladder operations on $D$ and showed that these operations didn’t affect the underlying permutation $w$ of $D$.

We will now introduce two special rc-graphs associated to any given $w$, and show that they are uniquely obtainable by a sequence of inverse chute or ladder moves.

**Definition 1.** For $w \in S_n$, set $D_{\text{bot}}(w) = \{(i, c) | c \leq m_i\}$, where $m_i$ is the number of $j > i$ such that $w_j < w_i$.

Notice that $D_{\text{bot}}(w)$ is ‘left-justified;’ identifying the rc-graph with a string diagram of the permutation $w$, the left side of the diagram is the ‘bottom,’ justifying the choice of name.

**Definition 2.** For $w \in S_n$, set $D_{\text{top}}(w) = \{(c, j) | c \leq n_j\}$, where $n_j$ is the number of $i$ such that $i < w_{j^{-1}}$ and $j > w_i$.

Notice that $D_{\text{top}}(w) = D_{\text{bot}}(w^{-1})$.

For example, let $w = [3, 1, 4, 6, 5, 2]$. Then $m_1 = 2$, as $w_1 = 3$ and 1 and 2 lie to the right of 3. Likewise, $m_2 = 0$, $m_3 = 1$, $m_4 = 2$, $m_5 = 1$, and $m_6 = 0$.

Then for the rc-graphs $D_{\text{bot}}$ and $D_{\text{top}}$ we respectively get:

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Set $\mathcal{C}(D)$ to be the set of diagrams obtainable from $D$ by Chute moves, and $\mathcal{L}(D)$ to be the set of diagrams obtainable by Ladder moves. Set $RC(w)$ to be the complete set of rc-graphs associated to $w$. Then we have the following result:

**Theorem 3.**

1. $D_{\text{top}}(w)$ does not admit an inverse chute move.

2. Any $D \in RC(w)$ such that $D \neq D_{\text{top}}(w)$ admits an inverse chute move.

3. $\mathcal{C}(D_{\text{top}}(w)) = RC(w) = \mathcal{L}(D_{\text{bot}}(w))$

4. $\sigma_w(x) = \sum_{D \in \mathcal{C}(D_{\text{top}}(w))} x^D = \sum_{D \in \mathcal{L}(D_{\text{bot}}(w))} x^D$, where $x^D = \prod_{(i,j) \in D} x_i$.

**Proof.**

1. Every column of $D_{\text{top}}(w)$ begins with an initial run of crossings, and then no more. As such, no column has an empty space above a crossing, and by the Lemma in the previous section, the diagram admits no inverse Chute moves.

2. For any $w' \in S_n$, take some $D \in RC(w)$ such that $D$ does not admit any inverse Chute move. Then column $j$ has some number $k$ of crossings gathered at the top, with $0 \leq k \leq n - j$. Then to construct such a diagram, we have $n$ choices for the first column, $n - 1$ for the second, and so on, yielding a total of $n!$ such diagrams. Since no two permutations can have matching diagrams, we have a one-to-one correspondence between rc-graphs admitting no inverse Chute moves and elements of the symmetric group. Then $D_{\text{top}}(w)$ is the unique rc-graph for $w$ admitting no inverse Chute moves.

3. Any rc-graph obtained from $D_{\text{top}}(w)$ by inverse Chute moves is in $RC(w)$ by the lemma of the last section. By the uniqueness of $D_{\text{top}}(w)$, we can conclude that $\mathcal{C}(D_{\text{top}}(w))$ is connected under the Chute move operations, and that $\mathcal{C}(D_{\text{top}}(w))$ is indeed all of $RC(w)$.

4. The final result follows immediately from the third part of this theorem and the combinatorial formula for the Schubert polynomials.

Because of the relationship between $D_{\text{bot}}(w)$ and $D_{\text{top}}(w)$, analogous results follow for $D_{\text{bot}}(w)$.
Example: Let’s compute $\sigma_{[1,4,3,2]}$. To write $D_{\text{bot}}(w)$, notice that $w_1 = 1$, and that nothing to the right of $w_1$ is smaller than 1. Then there are no crossings in the first row of $D_{\text{bot}}(w)$. Now $w_2 = 4$, and both 3 and 2 are less than 4, so the second row of $D_{\text{bot}}(w)$ has two crossings. Likewise, the third row contains one crossing, and the last row has no crossings. Since $D_{\text{bot}}(w)$ is ‘left-justified,’ we obtain for $D_{\text{bot}}(w)$ the following rc-graph:

```
1 2 3 4
1
2
3
4
```

Applying inverse chute moves to this rc-graph, we can obtain the following diagrams. The last is $D_{\text{top}}(w)$.

```
1 2 3 4
|   |   |
1  |   |
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4  |   |

1 2 3 4
|   |   |
1  |   |
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1 2 3 4
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3  |   |
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1 2 3 4
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```

Now, to compute $\sigma_{[1,4,3,2]}$ we take $x^D$ for each diagram $D$ and sum over the set of diagrams. For example, $x^{D_{\text{bot}}(w)} = x_2^2 x_3$. The resulting polynomial is $\sigma_{[1,4,3,2]}(x) = x_2^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^3 x_3 + x_1^2 x_2$, where the terms are here listed in the same order as the diagrams above.

Notice that if I dream up any old monomial, I can dream up quite a few rc-graphs that produces that monomial. Furthermore, $D_{\text{bot}}(w)$ arises from the largest reduced word for $w$ in reverse lexicographic order and largest compatible sequence in lexicographic ordering. Thus, given any monomial, I can produce an rc-graph that is the bottom-most rc-graph for some $w \in S_N$ for a sufficiently large $N$. Since lexicographic ordering is a total ordering of the monomial basis of $\mathbb{Z}[x_1, x_2, \ldots]$, the set of Schubert polynomials are expressible as an upper-triangular combination of monomials, with leading coefficient 1. We have proved the desired result:

**Theorem 4.** The set of Schubert polynomials $\sigma_w(x), w \in S_\infty$ form an integral basis of $\mathbb{Z}[x_1, x_2, \ldots]$.  

One can apply the same trick with $D_{\text{top}}(w)$, which produces a minimal monomial in reduced lexicographic order.

Next time we will prove Monk’s Formula, which gives a Pieri-like rule for multiplying the Schubert polynomials.