Lecture 17: Properties of rc-graphs and Monk's Rule

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The goal of this section is to use rc-graphs to show that the set of Schubert polynomials form a basis for $\mathbb{Z}[x_1, x_2, \ldots]$. Previously we established a combinatorial formula for the Schubert polynomials, which was a sum over reduced words a for an element $w \in S_n$ and *a-compatible sequences,* α. We also defined the rc-graph D associated to a pairing (a, α), defined the Chute and Ladder operations on D and showed that these operations didn't affect the underlying permutation w of D.

We will now introduce two special rc-graphs associated to any given w, and show that they are uniquely obtainable by a sequence of inverse chute or ladder moves.

Definition 1. *For $w \in S_n$, set $D_{bot}(w) = \{(i, c) | c \leq m_i\}$, where m_i is the number of $j > i$ such that $w_j < w_i$.*

Notice that $D_{bot}(w)$ is 'left-justified;' identifying the rc-graph with a string diagram of the permutation w, the left side of the diagram is the 'bottom,' justifying the choice of name.

Definition 2. *For $w \in S_n$, set $D_{top}(w) = \{(c, j) | c \leq n_j\}$, where n_j is the number of i such that $i < w_j^{-1}$ and $j > w_i$.*

Notice that $D_{top}(w) = D^t_{bot}(w^{-1})$.

For example, let $w = [3, 1, 4, 6, 5, 2]$. Then $m_1 = 2$, as $w_1 = 3$ and 1 and 2 lie to the right of 3. Likewise, $m_2 = 0, m_3 = 1, m_4 = 2, m_5 = 1$, and $m_6 = 0$. Then for the rc-graphs D_{bot} and D_{top} we respectively get:

	1	2	3	4	5	6
1	+	+				
2						
3	+					
4	+	+				
5	+					
6						

	1	2	3	4	5	6
1	+	+			+	
2		+				
3		+				
4		+				
5						
6						

Set $\mathcal{C}(D)$ to be the set of diagrams obtainable from D by Chute moves, and $\mathcal{L}(D)$ to be the set of diagrams obtainable by Ladder moves. Set $RC(w)$ to be the complete set of rc-graphs associated to w. Then we have the following result:

Theorem 3.

1. *$D_{top}(w)$ does not admit an inverse chute move.*

2. *Any $D \in RC(w)$ such that $D \neq D_{top}(w)$ admits an inverse chute move.*

3. *$\mathcal{C}(D_{top}(w)) = RC(w) = \mathcal{L}(D_{bot}(w))$*

4. *$\sigma_w(x) = \sum_{D\in\mathcal{C}(D_{top}(w))} x^D = \sum_{D\in\mathcal{L}(D_{bot}(w))} x^D$, where $x^D = \prod_{(i,j)\in D} x_i$.*

Proof.

1. Every column of $D_{top}(w)$ begins with an initial run of crossings, and then no more. As such, no column has an empty space above a crossing, and by the Lemma in the previous section, the diagram admits no inverse Chute moves.

2. For any $w' \in S_n$, take some $D \in RC(w)$ such that D does not admit any inverse Chute move. Then column j has some number k of crossings gathered at the top, with $0 \leq k \leq n - j$. Then to construct such a diagram, we have n choices for the first column, $n - 1$ for the second, and so on, yielding a total of $n!$ such diagrams. Since no two permutations can have matching diagrams, we have a one-to-one correspondence between rc-graphs admitting no inverse Chute moves and elements of the symmetric group. Then $D_{top}(w)$ is the unique rc-graph for w admitting no inverse Chute moves.

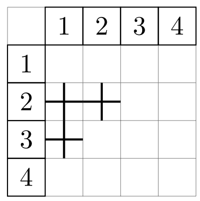
3. Any rc-graph obtained from $D_{top}(w)$ by inverse Chute moves is in $RC(w)$ by the lemma of the last section. By the uniqueness of $D_{top}(w)$, we can conclude that $\mathcal{C}(D_{top}(w))$ is connected under the Chute move operations, and that $\mathcal{C}(D_{top}(w))$ is indeed all of $RC(w)$.

4. The final result follows immediately from the third part of this theorem and the combinatorial formula for the Schubert polynomials.

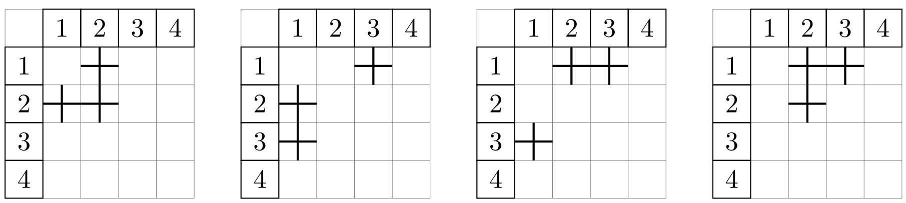
□

Because of the relationship between $D_{bot}(w)$ and $D_{top}(w)$, analagous results follow for $D_{bot}(w)$.

Example: Let's compute $\sigma_{[1,4,3,2]}$. To write $D_{bot}(w)$, notice that $w_1 = 1$, and that nothing to the right of w_1 is smaller than 1. Then there are no crossings in the first row of $D_{bot}(w)$. Now $w_2 = 4$, and both 3 and 2 are less than 4, so the second row of $D_{bot}(w)$ has two crossings. Likewise, the third row contains one crossing, and the last row has no crossings. Since $D_{bot}(w)$ is 'left-justified,' we obtain for $D_{bot}(w)$ the following rc-graph:



Applying inverse chute moves to this rc-graph, we can obtain the following diagrams. The last is $D_{top}(w)$.



Now, to compute $\sigma_{[1,4,3,2]}$ we take x^D for each diagram D and sum over the set of diagrams. For example, $x^{D_{bot}(w)} = x_2^2 x_3$. The resulting polynomial is $\sigma_{[1,4,3,2]}(x) = x_2^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^2 x^3 + x_1^2 x_2$, where the terms are here listed in the same order as the diagrams above.

Notice that if I dream up any old monomial, I can dream up quite a few rc-graphs that produces that monomial. Furthermore, $D_{bot}(w)$ arises from the largest reduced word for w in reverse lexicographic order and largest compatible sequence in lexicographic ordering. Thus, given any monomial, I can produce an rc-graph that is the bottom-most rc-graph for some $w \in S_N$ for a sufficiently large N. Since lexicographic ordering is a total ordering of the monomial basis of $\mathbb{Z}[x_1, x_2, \ldots]$, the set of Schubert polynomials are expressible as an upper-triangular combination of monomials, with leading coefficient 1. We have proved the desired result:

Theorem 4. *The set of Schubert polynomials $\sigma_w(x), w \in S_\infty$ form an integral basis of $\mathbb{Z}[x_1, x_2, \ldots]$.*

One can apply the same trick with $D_{top}(w)$, which produces a minimal monomial in reduced lexicographic order.

Next time we will prove Monk's Formula, which gives a Pieri-like rule for multiplying the Schubert polynomials.