LECTURE 19: STANLEY SYMMETRIC FUNCTIONS AND THE AFFINE SYMMETRIC GROUP

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1. Properties of Stanley Symmetric Functions

Note: This section is following the paper "Noncommutative Schur functions and their applications" by Fomin and Greene (Discrete Math 193, 1998 pg 179-200).

Theorem 1.1. *The Stanley symmetric function $F_w(x)$ can be written as*

$$F_w(x) = \sum_{\lambda} \langle s_{\lambda^t}(u) \cdot 1, w \rangle \, s_\lambda(x)$$

where

$$\langle s_{\lambda^t} \cdot 1, w \rangle = c_\lambda^w = |\{T \in SSYT(\lambda^t) \mid w(T) \cdot 1 = w\}|.$$

This theorem needs some explaining: $w(T)$ is the column reading word of the semi-standard Young tableau T read from bottom to top, left to right (in English notation). This word gives the indices of the product of the u's acting on the identity permutation 1. As an example, let

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$$

then $w(T) = 652131423$, and we would have $u_6 u_5 \ldots u_2 u_3 \cdot 1$. Recall that the u's are the generators of the nil-Coxeter algebra.

We still need to describe $s_{\lambda^t}(u) \cdot 1$ appearing in the theorem. These are the so called non-commutative Schur functions.

Definition 1.2. The non-commutative Schur functions are

$$s_\lambda(u) = s_\lambda(u_1, \ldots, u_n) = \sum_{T \in SSYT(\lambda)} u^T$$

where $u^T = \prod_i u_i$ with the indices taken by the reading word of T (that is, $w(T)$).

Example 1.3. Let $\lambda = (3, 2)$, and consider only 2 variables. The semi-standard Young tableaux we get are $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$. Thus we have

$$s_\lambda(u_1, u_2) = u_2 u_1 u_2 u_1 u_1 + u_2 u_1 u_2 u_1 u_2.$$

We next have the following theorem concerning non-commutative Schur functions.

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Theorem 1.4 (Fomin, Greene)**.** *Suppose u_i is a list of indeterminates satisfying*

$$\begin{aligned} u_i u_k u_j &= u_k u_i u_j && \text{for } i \le j < k, |i-k| \ge 2 \\ u_j u_i u_k &= u_j u_k u_i && \text{for } i < j \le k, |i-k| \ge 2 \\ (u_i + u_{i+1}) u_{i+1} u_i &= u_{i+1} u_i (u_i + u_{i+1}). \end{aligned}$$

Then the map $s_{\lambda/\mu} \mapsto s_{\lambda/\mu}(u)$ extends to a homomorphism from the algebra Λ_n of symmetric polynomials in n commuting variables to $\Lambda_n(u)$ generated by $s_{\lambda/\mu}(u)$.

Remarks:

(1) The first two relations are called the non-local Knuth relations.
(2) All the relations above hold for the nil-Coxeter algebra.
(3) The significance of this theorem is that all properties of usual Schur functions $s_{\lambda/\mu}$ hold for $s_{\lambda/\mu}(u)$. So we have $s_{\lambda/\mu}(u)$ commute with each other, $s_{\lambda/\mu}(u)$ span $\Lambda_n(u)$ as a $\mathbb{Z}$-module, and $s_{\lambda/\mu}(u)$ multiply according to Littlewood-Richardson rule.

Outline of Proof. (1) First, prove that the corresponding elementary functions in $\Lambda_n(u)$ commute: $e_j(u)e_k(u) = e_k(u)e_j(u)$ where $e_k(u) = \sum_{a_1 > a_2 > \cdots > a_k} u_{a_1} \ldots u_{a_k}$.

(2) Prove the Jacobi-Trudi identity in the non-commutative setting:

$$s_{\lambda/\mu}(u) = \det(e_{\lambda_i^t - \mu_j^t + j - i}(u))_{i,j=1}^n$$

using Gessel-Viennot paths.

□

We also have the non-commutative Cauchy identity

Theorem 1.5.

$$\prod_{i=1}^{m} \prod_{j=n}^{1} (1 + x_i u_j) = \sum_{\lambda} s_\lambda(x) s_{\lambda^t}(u).$$

Now we are in shape to prove the first theorem of the section.

Proof of theorem 1.1.

$$\begin{aligned} F_w(x) &= \langle \prod_{i=1}^{m} \prod_{j=n}^{1} (1 + x_i u_j) \cdot 1, w \rangle \\ &= \langle \sum_{\lambda} s_\lambda(x) s_{\lambda^t}(u) \cdot 1, w \rangle \\ &= \sum_{\lambda} \langle s_{\lambda^t}(u) \cdot 1, w \rangle \, s_\lambda(x). \end{aligned}$$

□

2. Affine Symmetric Group

Definition 2.1. The affine symmetric group $\tilde{S_n}$ for $n \ge 2$ is the group of bijections w on $\mathbb{Z}$ such that

(1) $w(i+n) = w(i) + n$ for all $i \in \mathbb{Z}$
(2) $\sum_{i=1}^n w(i) = \binom{n+1}{2}$.

The product in the group is function composition, and $w \in \tilde{S_n}$ is called an affine permutation.

Remark: $w \in \tilde{S_n}$ is uniquely specified by its values on $[n]$. This leads to the window notation $w = [w(1), w(2), \ldots, w(n)]$.

Example 2.2. If $u = [2, 1, -2, 0, 14]$ and $v = [15, -3, -2, 4, 1]$ in $\tilde{S_5}$, then $uv = [24, -4, -7, 0, 2]$.

We can specify generators for $\tilde{S_n}$. Let $\tilde{S_A} = S = \{s_o, s_1, \ldots, s_{n-1}\}$ where

$$s_i = [1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n] \text{ for } i = 1, 2, \ldots, n-1$$

and

$$s_0 = [0, 2, \ldots, n-1, n+1].$$

We can also consider what happens when $s_i \in S$ acts on $u \in \tilde{S_n}$ on the right. The claim is that us_i interchanges the entries in positions $i + kn$ and $(i+1) + kn$ for all $k \in \mathbb{Z}$ in u. In window notation this looks like

$$us_i = \begin{cases} [u(1), \ldots, u(i-1), u(i+1), u(i), u(i+2), \ldots, u(n)] & 1 \leq i \leq n-1 \\ [u(0), u(2), \ldots, u(n-1), u(n+1)] & i = 0 \end{cases}.$$