LECTURE 21: (K+1)-CORES AND K-BOUNDED PARTITIONS

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1. Reflections

$a, b \in \mathbb{Z}, a \not\equiv b \text{ mod } n$
$t_{a,b} = \prod_{r\in\mathbb{Z}}(a + rn, b + rn)$

Proposition 1.1. *The set of all reflections of* $\tilde{S}_n$ $\{t_{i,j+kn} | 1 \le i \le j \le n, k \in \mathbb{Z}\}$

Proposition 1.2. *Let* $u, v \in \tilde{S_n}$*, then TFAE.*
(1) $u \to v$ *is Bruhat order.*
(2) $\exists i, j \in \boldsymbol{Z}, i < j, i \not\equiv j$ *mod* n *such that* $u(i) < u(j)$ *and* $v = ut_{i,j}$

Proof. By the definition of Bruhat order (1)⇔(2) reduces to showing
if $v = ut_{i,j}$ (v is obtained from u by interchanging $u(i + rn)$ and $u(j + rn)\, \forall r \in \mathbb{Z}$)
then $\widetilde{inv}(v) > \widetilde{inv}(u)$ if $u(i) < u(j)$.
Proof is analogous to the proof that $\ell(v) = \widetilde{inv}(v)$. □

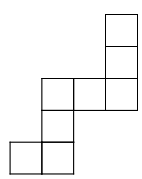
2. n-cores

From this point on $n = k + 1$.

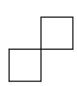
AIM : We want to establish a correspondence
k-tableaux ⟷ reduced words for Grassmannian affine permutations
shape/(k+1)-cores⟷ Grassmannian affine permutations
(shape/(k+1)-cores≅ k-bounded partitions)

Definition 2.1. An n-ribbon is connected skewshape λ/μ without 2×2 squares such that $|\lambda/\mu| = n$.

Example 2.2. (1)

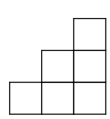


(2)



is not connected.

(3)



is not a ribbon.

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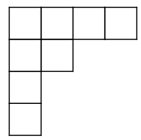
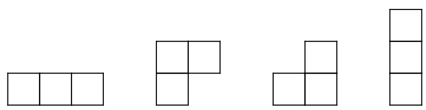
Definition 2.3. $\lambda \in P$ is an n-core if no n-ribbon can be removed from λ to obtain another partition.

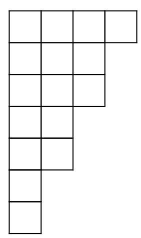
Example 2.4. (1) The following is a 3-core:



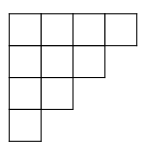
Possible three-ribbons to remove are:



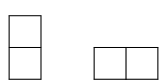
(2) The following is also a 3-core



(3) The next partition is a 2-core



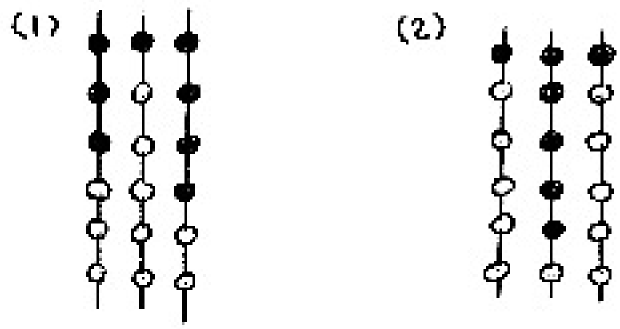
Possible 2-ribbons to remove are:



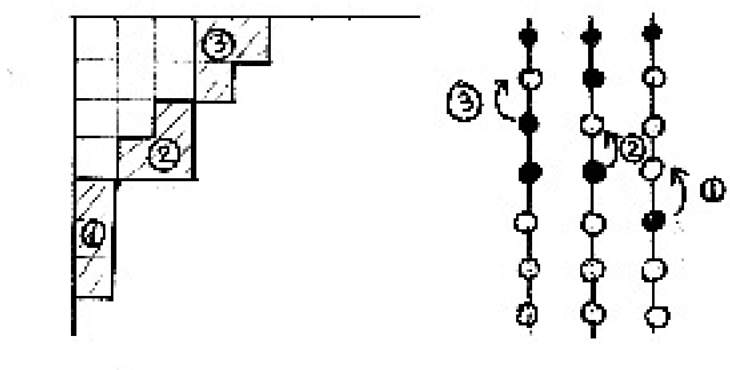
3. Abacus

Scan the boundary of the partition from right to left. Every $-$ (horizontal) step yields $\bullet$ and every $|$ (vertical) step yields $\circ$ on an n-strand abacus, where the beats are placed reading top to bottom left to right on the abacus with n-strands.

Example 3.1. n=3



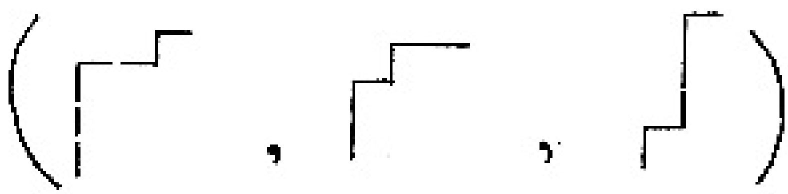
Example 3.2. (1) The following is not a 3-core

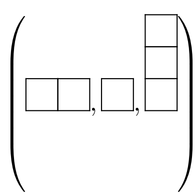


Note that removal of an an n-ribbon corresponds to moving a beat up in its strand. n-cores are those abacus configurations where all beats are at the top of their strands.

Definition 3.3. The vector of n partitions obtained by interpreting each strand of λ as a partition is called the n-quotient of λ.

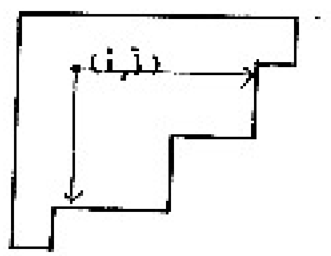
Example 3.4. The 3-quotient of the previous example is



$=$ 

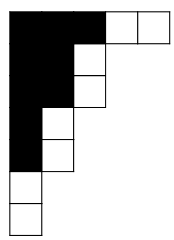
4. Bijection between (k+1) cores and k-bounded partitions

Definition 4.1. λ is a k-bounded partition if $\lambda_1 \leq k$. The hook length of the cell (i, j), where i denotes its row and j its column index, is the length of its hooks as defined in the picture:

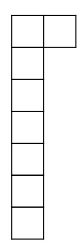


Definition 4.2. For the bijection from $(k+1)$-cores to k-bounded partitions, remove all cells with hooks greater than $k+1$ (note that $(k+1)$-cores have no boxes with hook length $k+1$).

Example 4.3. In the following 3-core we blacked out all boxes with hook $> k+1$:



Sliding the parts to the left yields the 2-bounded partition



5. Weak k-tableaux

Definition 5.1. Let $c = (i, j) \in \lambda$ be a cell in a partition. The content of $c = (i, j)$ is $j - i$. Its (k+1)-residue is $j - i$ mod k+1.

Next time we will use this to defined weak k-tableaux.