LECTURE 25: AFFINE STANLEY SYMMETRIC FUNCTIONS

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1. Review of previous material:

Recall that we defined the Stanley symmetric functions in a previous lecture as:

$$F_\omega(x) = \lim_{s\to\infty} \sigma_{1^s \times \omega} = \langle H(x_1) H(x_2) \ldots \cdot 1, \omega \rangle$$

where $H(x) = (1 + xu_{n-1})(1 + xu_{n-2}) \ldots (1 + xu_1)$. This can be rewritten as

$$F_\omega(x) = \sum_{\substack{a=(a_1,\ldots,a_l)\\ \text{compositions}}} \langle A_{a_t}(u), \ldots, A_{a_1}(u) \cdot 1, \omega \rangle \cdot x_1^{a_1} \cdots x_t^{a_t}$$

where $A_k(u) = \sum_{b_1 > b_2 > \cdots > b_k} u_{b_1} \cdots u_{b_k}$.

2. Affine Stanley Symmetric functions.

The affine nil-Coxeter algebra U_n is generated by $u_0, u_1, \ldots, u_{n-1}$ with the relations:

$u^2 = 0$

$u_i u_j = u_j u_i$ if $|i - j| \neq \pm 1$ mod n

$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ with indices reduced mod n.

Definition 2.1. Let $\underline{a} = a_1 \ldots a_k$ be a word such that $a_i \in \{0, 1, \ldots, n-1\}$ and $a_i \neq a_j$ when $i \neq j$. We say that $\underline{a}$ is cyclically decreasing if for all i such that i and $i+1$ mod n appear in $\underline{a}$, $i+1$ appears before i in $\underline{a}$. In particular, $\underline{a}$ cannot be cyclically decreasing if no indices are missing.

$u \in U_n$ is cyclically decreasing if $u = u_{a_1} \cdots u_{a_k}$ with $\underline{a} = a_1 \ldots a_k$ cyclically decreasing.

Example 2.2. Take $n = 6$. Then 32105 is cyclically decreasing since $2 < 3$, $1 < 2$, $0 < 1$ and $5 < 6$. But 432105 is not cyclically decreasing, since every index is included.

Remark 2.3. A cyclically decreasing element is completely determined by the set $A = \{a_1, \ldots, a_k\} \subset [0, n-1]$, and we can write u_A for this u.

Definition 2.4. $A_k(u) \in U_n$ for $0 \leq k \leq n-1$ is defined as follows:

$$A_k(u) = \sum_A u_A$$

where the sum is over all $A \in \binom{[0, n-1]}{k}$, that is, all k-subsets of $[0, n-1]$.

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Definition 2.5. Given $\omega \in \tilde{S}_n$, the affine Stanley symmetric functions are defined as:

$$\tilde{\mathcal{F}}_\omega(x) = \sum_{\substack{a=(a_1,\ldots,a_l)\\ \text{compositions of } \ell(w)}} \langle A_{a_k}(u)\cdots A_{a_1}(u)\cdot 1, \omega\rangle \cdot x_1^{a_1}\cdots x_t^{a_k}$$

Proposition 2.6.

(1) $[x_1 \cdots x_{\ell(w)}]\tilde{\mathcal{F}}_\omega = \#$ *reduced words for* w

(2) *for* $\omega \in S_n \subset \tilde{S}_n$ *:*

$$\tilde{\mathcal{F}}_\omega = \mathcal{F}_\omega$$

Theorem 2.7. $\tilde{\mathcal{F}}_\omega(x)$ *are symmetric functions.*

Sketch of proof. One needs to prove that the $A_k(u)$ commute (see T.Lam [1]). □

(Another case, $u_i^2 = u_i$ instead of $u_i^2 = 0$, leads to affine stable Grothendiek polynomials related to the K-theory for the affine Grassmannian, see [3]. They are nonhomogenous symmetric functions).

Definition 2.8. Define the affine Schur functions (or dual k-Schur functions)

$$\tilde{F}_\lambda(x) = \tilde{F}_{\mathcal{C}^{-1}(b^{-1}(\lambda))}(x),$$

where λ is a k-bounded partition; (use bijection from cores to affine Grassmannian elements).

Theorem 2.9. *The affine Schur functions* $\{\tilde{F}_\mu | \mu \in \mathcal{P}, \mu_1 \leq k\}$ *form a basis for* $\Lambda^{(n)} = \mathbb{C}\langle m_\lambda | \lambda \in \mathcal{P}, \lambda_1 \leq k\rangle$.

Conjecture 2.10. *The expansion*

$$\tilde{F}_w(x) = \sum_\lambda a_{w\lambda}\tilde{F}_\lambda(x), \quad a_{w\lambda} \in \mathbb{N}$$

References

[1] T. Lam, *Affine Stanley Symmetric Functions*, arXiv:math/0501335v1 [math.CO].

[2] T. Lam, *Schubert Polynomials for the Affine Grassmannian*, arXiv:math/0603125v2 [math.CO].

[3] T. Lam, A. Schilling,M. Shizomono, *K-theory Schubert Calculus of the Affine Grassmannian*, arXiv:0901.1506v1 [math.CO]