LECTURE 4: STRONG EXCHANGE PROPERTY

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1. Reduced Words and Length

Let (W, S) be a Coxeter system. Recall the definition of *length* of an element $\omega \in W$:

Definition 1.1. *The length $\ell(\omega)$ of $\omega \in W$ is the minimal $k \in \mathbb{Z}_{\geq 0}$ such that $\omega = s_1 s_2 \cdots s_k$ is an expression of ω in terms of the generators $s_i \in S$. Any word $s_1 \cdots s_{\ell(\omega)}$ such that $\omega = s_1 \cdots s_{\ell(\omega)}$ is called a* reduced *word.*

Lemma 1.2. *The map $\epsilon : s \to -1 \; \forall s \in S$ extends to a group homomorphism $\epsilon : W \to \{\pm 1\}$.*

Proof. Just need to check that any two words for an element $\omega \in W$ differ by an even number of generators. This follows from the Coxeter relations. □

Proposition 1.3. *For all $\omega, \omega' \in W$, and $s \in S$:*
1. $\epsilon(\omega) = (-1)^{\ell(w)}$
2. $\ell(\omega'\omega) \cong \ell(\omega') + \ell(\omega) \mod 2$
3. $\ell(s\omega) = \ell(\omega) \pm 1$
4. $\ell(\omega) = \ell(\omega^{-1})$
5. $|\ell(\omega') - \ell(\omega)| \leq \ell(\omega'\omega) \leq \ell(\omega') + \ell(\omega)$.

Proof. (1) - (3) follow from above lemma. For (4), suppose $\ell(\omega^{-1}) < \ell(\omega)$, and say $\omega^{-1} = s_1 \cdots s_k$. Then we can write $\omega = s_k \cdots s_1$, but we assumed all minimal words for ω had more than k generators so we get a contradiction. Interchange ω and ω^{-1} and (4) follows. The second inequality of (5) is clear – we can just concatenate a reduced word for ω' with a reduced word for ω to get a word for $\omega'\omega$. The first inequality follows from the Coxeter relations for W. The only way to reduce the length of a word is through the relation $s^2 = 1$. Given reduced words for ω' and ω, one can check that the maximum number of generators that can be canceled in $\omega'\omega$ is $\min(\ell(\omega), \ell(\omega'))$. □

Remark 1.4. $A := \{\omega \in W \mid \ell(\omega) \equiv 0 \mod 2\}$ *is a subgroup of W called the alternating subgroup (also called the rotation subgroup).*

2. Strong Exchange Property

Theorem 2.1 (Strong Exchange Property)**.** *Let (W, S) be a Coxeter system, let $T = \{\omega s \omega^{-1} \mid \omega \in W\}$ be the set of reflections of W and let $\omega = s_1 \cdots s_k \in W$, $s_i \in S$, $t \in T$. If $\ell(t\omega) < \ell(\omega)$, then $t\omega = s_1 \cdots \hat{s}_i \cdots s_k$ for some $1 \leq i \leq k$.*

Before proving the theorem, we recall several definitions from a previous lecture:

- $\hat{T}(s_1 \cdots s_k) = (t_1, t_2, \ldots, t_k)$, where $t_i = s_1 s_2 \cdots s_{i-1} s_i s_{i-1} \cdots s_2 s_1$.

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- $n(s_1 \cdots s_k; t) =$ the number of times t appears in $\hat{T}(s_1 \cdots s_k)$.
- $\eta(s;t) = \begin{cases} 1 & \text{if } s = t \\ -1 & \text{if } s \neq t. \end{cases}$
- $R := T \times \{\pm 1\}$
- $\pi_s : R \to R$, $\pi_s(t, \epsilon) := (sts, \epsilon\eta(s;t))$. The map $s \to \pi_s$ can be extended uniquely to an injective homomorphism $\omega \to \pi_\omega$ from W to $S(R)$, the group of permutations of R.

Note that we can extend the definition of $\eta(s;t)$ to all of W by $\eta(\omega; t) := (-1)^{n(s_1 s_2 \cdots s_k; t)}$, where $s_1 s_2 \cdots s_k$ is an arbitrary expression for ω. The parity of $n(s_1 s_2 \cdots s_k; t)$ depends only on ω and t (see proof that $s \to \pi_s$ extends uniquely to $\omega \to \pi_\omega$ from last lecture) and so $\eta(\omega; t)$ is well-defined.

Proof of Theorem 2.1.
Claim: $\ell(tw) < \ell(w) \Longleftrightarrow \eta(w,t) = -1$

Proof. *(of Claim)*

" $\Longleftarrow$ " assume $\eta(w,t) = -1$ $(*)$
$w = s_1\prime ... s_d\prime$ is a reduced expression for w
$n(s_1\prime ... s_d\prime; t)$ is odd by $(*)$
$\Longrightarrow t = s_1\prime ... s_i\prime ... s_1\prime$ for some $1 \leq i \leq d$
$\Longrightarrow \ell(tw) = \ell(s_1\prime ... \widehat{s_i\prime} ... s_d\prime) < d = \ell(w)$
" $\Longrightarrow$ " Asssume $\eta(w,t) = 1$

$$\begin{aligned} \Pi_{(tw)^{-1}}(t,\varepsilon) &= \Pi_{w^{-1}}\Pi_t(t,\varepsilon) \\ &= \Pi_{w^{-1}}(t,-\varepsilon) \\ &= (w^{-1}tw, -\varepsilon\eta(w,t)) \\ &= (w^{-1}tw, -\varepsilon) \end{aligned}$$

where from lines 2 to 3 we are using the Theorem from the last lecture.
$\Longrightarrow \eta(tw;t) = -1$ (Reading off definition)
$\Longrightarrow$ by " $\Longleftarrow$ " direction $\ell(w) = \ell(ttw) < \ell(tw)$
Now $\ell(tw) < \ell(w) \Longrightarrow \eta(w,t) = -1$
$\Longrightarrow n(s_1 ... s_k; t)$ is odd
$\Longrightarrow t = s_1 ... s_i ... s_1$ for some $1 \leq i \leq k$
$\Longrightarrow tw = s_1 ... \widehat{s_i} ... s_k$ □

Corollary 2.2. *$(*)$*

$w = s_1 ... s_k$ a reduced word, $t \epsilon T$
T.F.A.E.:
(1) $\ell(tw) < \ell(w)$
(2) $tw = s_1 ... \widehat{s_i} ... s_k$ for some $1 \leq i \leq k$
(3) $t = s_1 s_2 ... s_i ... s_2 s_1$
Furthermore i in (2) and (3) are uniquely determined.

Proof.

(1) $\Longrightarrow$ (2) by Strong Exchange Property
(2) $\Longrightarrow$ (1) is obvious
(3) $\Longrightarrow$ (2) easy calculation

(2) $tw = s_1 ... \widehat{s_i} ... s_k$
$\Longrightarrow ts_1 ... s_i = s_1 ... s_{i-1}$
$\Longrightarrow t = s_1 ... s_i ... s_1$
Uniqueness of i follows from the lemma last lecture which said that if $w = s_1 ... s_k$ is reduced, then all t_i are distinct.

Definition 2.3.

$T_L(w) := \{t \epsilon T | \ell(tw) < \ell(w)\}$
$T_R(w) := \{t \epsilon T | \ell(wt) < \ell(w)\}$, note $T_R(w) = T_L(w^{-1})$
$D_L(w) = T_L(w) \cap S$ are the left descents
$D_R(w) = T_R(w) \cap S$ are the right descents

Corollary 2.4. $|T_L(w)| = \ell(w)$

Proof.

Let $w = s_1 ... s_k$ with $k = \ell(w)$. Then by Corollary $*$,
$T_L(w) = \{s_1 ... s_i ... s_1 | 1 \leq i \leq k\}$
Since $s_1 ... s_k$ is reduced, all $t_i = s_1 ... s_i ... s_1$ are distinct.

Corollary 2.5. *$\forall s \epsilon S$ and $w \epsilon W$*

(1) $s \epsilon D_L(w) \Longleftrightarrow$ some reduced expression for w begins with s
(2) $s \epsilon D_R(w) \Longleftrightarrow$ some reduced expression for w ends with s

Proof.

" $\Longleftarrow$ " clear
" $\Longrightarrow$ " By Corollary $*, \ell(tw) < \ell(w) \Longleftrightarrow tw = s_1 ... \widehat{s_i} ... s_k$
If $s = t$, then $w = stw = ss_1 ... \widehat{s_i} ... s_k$