LECTURE 5: CHARACTERIZATION THEOREM AND EXAMPLES

MIHAELA IFRIM AND BRANDON BARRETTE

1. Strong Exchange Property

We now review the Strong Exchange Property Theorem from last lecture.

Theorem 1.1. *Let $w = s_1 s_2 \dots s_k$ be a reduced expression for $w \in W$ with $s_i \in S$ and let $t \in T$. Then $\ell(tw) < \ell(w)$ implies that:*

$$tw = s_1 \cdots \hat{s}_i \cdots s_k \qquad \text{for some } 1 \le i \le k. \tag{1.1}$$

Corollary 1.2. *Let $w = s_1 \cdots s_k$ be a reduced word and let $t \in T$. Then the following are equivalent:*
(1) $\ell(tw) < \ell(w)$;
(2) $tw = s_1 \cdots \hat{s}_i \cdots s_k$ for some i;
(3) $t = s_1 s_2 \cdots s_i \cdots s_2 s_1$.

Proposition 1.3. *Deletion Property*
Let $w = s_1 \cdots s_k$ be such that $\ell(w) < k$.
Then $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$ for some $1 \le i < j \le k$.

Proof. We choose i maximal such that $s_i s_{i+1} \cdots s_k$ is not reduced and therefore $\ell(s_i \cdots s_k) < \ell(s_{i+1} \cdots s_k)$. By the Strong Exchange Property we obtain:

$$s_i \cdots s_k = s_{i+1} \cdots \hat{s}_j \cdots s_k \quad \text{for some } 1 < j \le k.$$

Using the equality above we obtain:

$$w = s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$$

which ends our proof. □

Corollary 1.4. *The following properties hold:*
(1) Any word $w = s_1 \cdots s_k$ contains a reduced word as a subword by deleting an even number of letters.
(2) Suppose that $s_1 \cdots s_k = s'_1 \cdots s'_k$ and also suppose that both are reduced. Then $\Rightarrow \{s_1, \dots, s_k\} = \{s'_1, \dots, s'_k\}$.
(3) S is a minimal generating set for W .

Proof. (1) follows from Deletion Property.
(2) Suppose $\exists\, s_j$ which is not included in the set $I := \{s'_1, \dots, s'_k\}$. Here we choose j minimal with the property just mentioned. By Corollary 1.2, if $t = s_1 \cdots s_j \cdots s_1$ then there must exists an i such that

$$s_1 \cdots s_j \cdots s_1 = s'_1 \cdots s'_i \cdots s'_1$$

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for some i.
Therefore $s_j = (s_{j-1} \cdots s_1)(s'_1 \cdots s'_i \cdots s'_1)(s_1 \cdots s_{j-1})$ - where all are letters in I.
Using the Deletion Property we can find a reduced subword of the right-hand side, but this will give us:

$$s_j = s'_a \in I$$

which is a contradiction with the assumption that s_j is not in I.
(3) Follows from (2) since no element $s \in S$ can be written as a product of other elements in S. $\square$

2. Characterization of Coxeter groups

We will assume that W is an arbitrary group. Let $S \subseteq W$ be a generating set such that $s^2 = e, \ \forall s \in S$. Therefore the concept of length, $\ell(w)$, where $w \in W$ still makes sense and the concept of reduced expressions also still makes sense.

In this new context, we say that the system (W, S) has the Exchange or Deletion property if the following hold:

The Exchange Property
Let $w = s_1 \cdots s_k$ be reduced, and let $s \in S$. Then $\ell(sw) < \ell(w) \Rightarrow sw = s_i \cdots \widehat{s_i} \cdots s_k$ for some i, $1 \leq i < j < k$.

The Deletion Property
If $w = s_1 \cdots s_k$, then $\ell(w) < k \Rightarrow w = s_i \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_k$ for $1 \leq i < j < k$.

Theorem 2.1. *Characterization Theorem*
Let W be a group group and let $S \subseteq W$ be a generating set with $s^2 = e \ \forall s \in S$.
Then the following are equivalent:
(1) (W, S) is a Coxeter system.
(2) W satisfies the Exchange Property.
(3) W satisfies the Deletion Property.

Proof. The proof will be presented in the next lecture. $\square$

Now let's look at the following example:

Example 2.2. S_n is the well known group of permutations of $[n]$. S_n is generated by $S = \{s_1, \cdots, s_{n-1}\}$ where $s_i = (i, i+1)$. In one line notation, we have $s_i = [1, \cdots, i-1, i+1, i, \cdots, n]$. Recall that $s_i^2 = e$.

Fixing $x \in S_n$ we recall that:

<u>*Right action by s_i*</u>:
Then xs_i is obtained from x by interchanging the positions of $x(i)$ and $x(i+1)$. For example we have $[31524]s_3 = [31254]$.

<u>*Left action by s_i*</u>:
Then $s_i x$ is contained from x by interchanging the values i and $i+1$.
A numerical example is the following $s_3[31524] = [41523]$and this shows that S generates S_n.

Definition 2.3. The inversion number of $x \in S_n$ is given by the following expression:

$$\mathrm{inv}(x) = \mid \{(i,j) \mid i < j,\ x(i) > x(j)\} \mid .$$

Looking at the definition it is easy to see that the following lemma holds:

Lemma 2.4. *The following equality holds:*

$$\mathrm{inv}(xs_i) = \mathrm{inv}(x) + \begin{cases} 1 & \text{if } x(i) < x(i+1) \\ -1 & \text{if } x(i) > x(i+1) \end{cases}$$

The property that we will prove now shows a very useful relation between the length of a word and the number of inversions of the word.

Proposition 2.5. *We have the following relation:*
$\ell(x) = \mathrm{inv}(x),\ \forall x \in S_n$.

Proof. We know that we have $\ell(e) = \mathrm{inv}(e)$. Then by the Lemma 2.4 we obtain that $\mathrm{inv}(x) \leq \ell(x)$.

Claim. $\ell(x) \leq \mathrm{inv}(x)$.

Proof. (of the claim) Since $\mathrm{inv}(x) = 0 \Rightarrow x = e \Rightarrow \ell(e) = 0$. Hence the claim is true for $\mathrm{inv}(x) = 0$. We proceed by induction on $\mathrm{inv}(x)$. Let $x \in S_n$ be such that $\mathrm{inv}(x) = k+1$. Then $x \neq e \Rightarrow \exists s \in S$ such that $\mathrm{inv}(xs) = k$. By the induction hyphothesis $\ell(xs) \leq k \Rightarrow \ell(x) \leq k+1 = \mathrm{inv}(x)$.This finishes the proof. □

□

We recall from our previous lectures that the descent set $D_R(x) = \{s \in S \mid \ell(xs) < \ell(x)\}$.

Proposition 2.6. *For S_n we have $D_R(x) = \{s_i \in S \mid \text{such that } x(i) > x(i+1)\}$. This implies that the definition of $D_R(x)$ that we wrote above is the same with the notion we just stated in the statement of the proposition.*

Proof. By the Proposition 2.5 we have:

$$D_R(x) = \{s \in S \mid \mathrm{inv}(xs) < \mathrm{inv}(x)\} = \{s_i \in S \mid \text{such that } x(i) > x(i+1)\}.$$

□

Proposition 2.7. *Using the Characterization Theorem we can prove that (S_n, S) is a Coxeter system of type A_{n-1}.*

The proof will be given in the next lecture.