LECTURE 6: PROOF OF CHARACTERIZATION THEOREM

BRANDON BARRETTE AND MIHAELA IFRIM

1. Characterization Theorem

Theorem 1.1. *(Characterization Theorem) Let W be a group and $S \subset W$ a generating set such that $s^2 = e$ $\forall s \in S$. Then the following are equivalent:*

(1) *(W, S) is a Coxeter system*
(2) *W has the Exchange Property*
(3) *W has the Deletion Property*

Proof. Proof will follow after some propositions and corollaries. □

Proposition 1.2. *(S_n, S) is a Coxeter system of type A_{n-1}.*

Proof. By the Characterization Theorem, it suffices to show that (S_n, S) satisfies the Exchange property. Notice the following properties show that S_n is of type A_{n-1}:

$$s_i s_j = s_j s_i \text{ if } |i - j| > 1$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$
$$s_i^2 = e$$

Next, suppose $w = s_{i_1} \dots s_{i_k}$ is a reduced word such that

$$\ell(s_{i_1} \dots s_{i_k} s_i) < \ell(s_{i_1} \dots s_{i_k}) \qquad (*)$$

Then, if we can prove this claim, we will have proven that (S_n, S) satisfies the Exchange property.

Claim. *$s_{i_1} \dots s_{i_k} s_i = s_{i_1} \dots \hat{s_{i_j}} \dots s_{i_k}$ for some $1 \leq j \leq k$.*

Proof. Let $a = w(i+1)$ and $b = w(i)$. We proved last lecture that $\ell(y) = \text{inv}(y)$ $\forall y \in S_n$. Then $(*)$ implies that $b < a$ and a is to the left of b in e in one line notation and a is to the right of b in w in one line notation. This implies that $\exists j$ such that a is to the left of b in $s_{i_1} \dots s_{i_{j-1}}$, and a is to the right of b in $s_{i_1} \dots s_{i_j}$. This implies, in one line notation, that $s_{i_1} \dots s_{i_k}$ is the same as $s_{i_1} \dots \hat{s_{i_j}} \dots s_{i_k}$ except that a and b are interchanged. This completes the proof of the claim. □

Therefore, since the claim is proven, (S_n, S) satisfies the Exchange property. □

Proof. (of the Characterization Theorem)
(1) $\implies$ (2) This is a special case of the strong exchange property.
(2) $\implies$ (3) This was already proved last lecture.
(3) $\implies$ (2) Suppose that $\ell(ss_1 \dots s_k) \leq \ell(s_1 \dots s_k) = k$. This means that $w = s_1 \dots s_k$ is reduced. By the Deletion property, two letters can be deleted from $ss_1 \cdots s_k$ to obtain an expression for sw. We have two cases:

Date: January 16, 2009.

Case 1:
Suppose s is not deleted, then $ss_1 \dots s_k = ss_1 \dots \hat{s_i} \dots \hat{s_j} \dots s_k$. But then $s_1 \dots s_k = w = s_1 \dots \hat{s_i} \dots \hat{s_j} \dots s_k$. This implies $\ell(w) < k$, which is a contradiction since $w = s_1 \dots s_k$ is already reduced.

Case 2:
Suppose s is deleted, therefore $sw = s_1 \dots \hat{s_i} \dots s_k$ for some $1 \leq i \leq k$
Therefore the Exchange property is satisfied.

(2) $\implies$ (1) Suppose (W, S) has the Exchange property. Let $s_1 \dots s_r = e$ be a relation in W, we need to show that this follows from a Coxeter relation, and thus (W, S) will be Coxeter system.

By the Deletion property (since (2) $\Leftrightarrow$ (3)) $r = 2k$ which implies the relation is equivalent to $s_1 \dots s_k = s'_1 \dots s'_k$.

Claim. *Any relation*

$$s_1 \dots s_k = s'_1 \dots s'_k \qquad (**)$$

is a consequence of pairwise relations $(ss')^{m(s,s')} = e$.

Before we begin the proof of the claim, it is important to understand some terminology. We say a relation is **fine** if this claim holds.

Proof. (of claim) Perform induction on k.
Show true for $k = 1$
Here $s = s'$ therefore $s^2 = e$, therefore true by assumptions.
Assume true for $k - 1$
That is, $s_1 \dots s_{k-1} = s'_1 \dots s'_{k-1}$
Prove true for k
There are two cases to consider here:

Case 1: $s_1 \dots s_k$ is not reduced.
Then, this implies that $\exists\ 1 \leq i \leq k$ such that $s_{i+1} \dots s_k$ is reduced, but $s_i \dots s_k$ is not reduced, thus by the Exchange property, we have:
$s_{i+1} \dots s_k = s_i s_{i+1} \dots \hat{s_j} \dots s_k$ for some $i < j \leq k$. Since the length $< k$, this expression is fine. Therefore:

$$s_1 \dots s_k = s'_1 \dots s'_k \text{ becomes } s_1 \dots s_i s_i s_{i+1} \dots \hat{s_j} \dots s_k = s'_1 \dots s'_k$$

is also fine since length $< k$.

Case 2: $s_1 \dots s_k$ is reduced.
Then, WLOG, we can assume that $s_1 \neq s'_1$ since otherwise $(**)$ reduces to a relation of length $< k$.

By the Exchange property:

$$s_1 \dots s_i = s'_1 s_1 \dots s_{i-1} \text{ for some } 1 \leq i \leq k \quad (\dagger)$$

Which implies

$$s_1 \dots \hat{s_i} \dots s_k = s'_2 \dots s'_k \text{ for some } 1 \leq i \leq k$$

Which is fine by induction.

If $\mathbf{i} < \mathbf{k}$, then † is also fine since $s'_1 \dots s'_k = s'_1 s_1 \dots \hat{s}_i \dots s_k$, which is fine, and implies $s_1 \dots s_k = s'_1 \dots s'_k$ by †.

If $\mathbf{i} = \mathbf{k}$, then $s_1 \dots s_{k-1} = s'_2 \dots s'_k$, which is fine since the length $< k$. This also implies $s'_1 s_1 \dots s_{k-1} = s'_1 \dots s'_k$ which is fine by multiplying by s'_1.

Now, we need to show $s'_1 s_1 \dots s_{k-1} = s_1 \dots s_k$ is fine.

To do this, repeat the argument with this new relation. One of two things will happen. Either, the fineness will be settles by Case 1, or we will end up in Case 2 and the relation will reduce to:

$$s_1 s'_1 s_1 \dots s_{k-2} = s'_1 s_1 \dots s_{k-1}$$

Is this fine? To answer the question, we repeat the argument yet again. If we fall in Case 1, we are done, otherwise we fall in Case 2 and again reduce the relation. Repeating this we will get the following relation:

$$s_1 s'_1 s_1 s'_1 \dots = s'_1 s_1 s'_1 s_1 \dots$$

which is a Coxeter relation.

Therefore by mathematical induction, our claim is proven.

□

Since our claim holds, we have that (W, S) is a Coxeter system. This completes the proof of the Characterization Theorem.

□