LECTURE 7: BRUHAT ORDER

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In this lecture, as before, we let (W, S) be a Coxeter system and T be the set of all conjugates:

$$T = \{wsw^{-1} | s \in S, w \in W\}.$$

1. Bruhat Order

Definition 1.1. Let u, v be two elements in W

(1) $u \xrightarrow{t} w$ means $ut = w$ for $t \in T$ and $\ell(u) < \ell(w)$
(2) $u \to w$ means $u \xrightarrow{t} w$ for some $t \in T$
(3) $u \leq w$ means that there exist $u_i \in W$ such that

$$u = u_0 \to u_1 \to u_2 \to \cdots \to u_k = w.$$

The Bruhat graph is a directed graph whose nodes are elements of W and whose edges are given by condition (2) of the previous definition. Also Bruhat order is the partial order given by condition (3).

Remark 1.2. From the definition it is clear that

(1) $u < v$ implies that $\ell(u) < \ell(w)$
(2) $u < ut$ if and only if $\ell(u) < \ell(ut)$, with $u \in W$ and $t \in T$.
(3) $e \leq w \; \forall w \in W$.

The last remark is true because if $w = s_1 s_2 \cdots s_k$ is a reduced expression for w, then we have the sequence of steps

$$e \to s_1 \to s_1 s_2 \to \cdots \to s_1 s_2 \cdots s_k = w$$

Remark 1.3. The condition $w = ut$ in definition 1.1 can be replaced by $w = t'u$ for $t' \in T$ and $t' = utu^{-1} \in T$.

[Bruhat order in S_n] Consider the system (S_n, S) where

$$S = \{s_1, \ldots, s_{n-1}\}$$

and $s_i = (i, i+1)$. We observe that

$$xs_ix^{-1} = (x(i), x(i+1)),$$

for $x \in S_n$, $s_i \in S$. Thus the reflection set is the set of all transpositions

$$T = \{(a, b) | 1 \leq a < b \leq n\}.$$

Since reflections in S_n are transpositions (a, b) and the length of a permutation equals its inversion number, $x \xrightarrow{(a,b)} y$ means that one moves from the permutation

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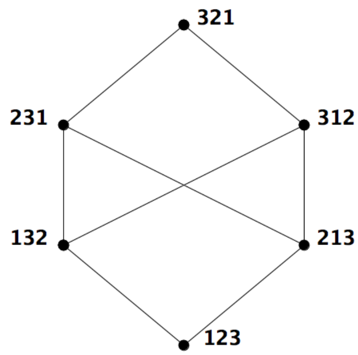


FIGURE 1. Bruhat Order for S_3

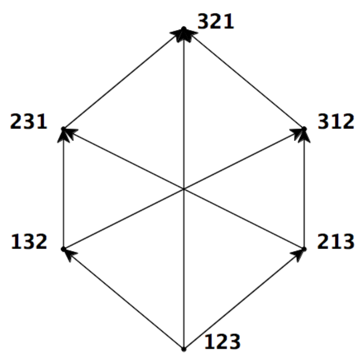


FIGURE 2. Bruhat Graph for S_3

$x = [x_1, \ldots, x_a, \ldots, x_b, \ldots, x_n]$ to the permutation $y = [x_1, \ldots, x_b, \ldots, x_a, \ldots, x_n]$ obtained by switching x_a and x_b, where $a < b$ and $x_a < x_b$. For example

$$21543 \xrightarrow{(2,5)} 23541.$$

From this construction we naturally get the Bruhat graph for S_n, a directed graph with edges between x and y if $x < y$. We also have a Hasse diagram for the Bruhat order on S_n obtained from the Bruhat graph by relaxing directedness and keeping edges corresponding to covering relations. Figure 1 shows the Bruhat order for S_3 and Figure 2 shows the Bruhat graph for S_3.

Lemma 1.4. *Let $x, y \in S_n$. Then, x is covered by y in the Bruhat order if and only if $y = x(a, b)$ for some a and b, $a < b$, such that $x(a) < x(b)$ and there does not exist c, $a < c < b$, such that $x(a) < x(c) < x(b)$.*

Proof. ($\Leftarrow$) If $y = x(a, b)$ with the stated conditions, then $\text{inv}(y) = \text{inv}(x) + 1$, thus we have a Bruhat covering relation.

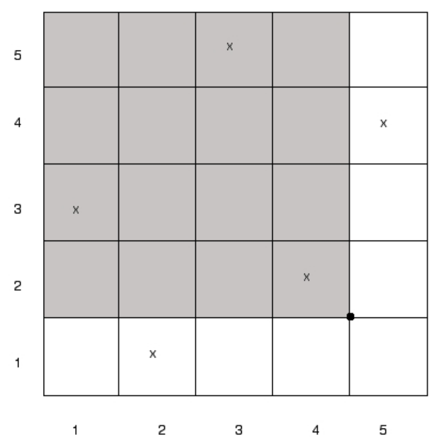
($\Rightarrow$) Conversely, suppose that $y = x(a, b)$, $a < b$, and $\text{inv}(y) > \text{inv}(x)$. Then $x(a) < x(b)$. If $x(a) < x(c) < x(b)$ for some a, b and c, $a < c < b$, then $x < x(a, c) < y$, so $x < y$ is not a Bruhat covering. □

Recall that Bruhat order is a partial order; two elements not necessarily comparable. This leads to the natural question: what are the necessary and sufficient conditions for determining when two elements in S_n are comparable in Bruhat order ? For example, how can we determine if $x = 368475912$ and $y = 694287531$ are comparable in Bruhat order ?

Let $x \in S_n$, and consider the collection of points in the square $[n] \times [n]$ given by $(i, x(i))$. We define the number of dots north-west of (i, j) in the diagram as follows

$$x[i, j] = |\{a \in [i] : x(a) \geq j\}| \tag{1.1}$$

For example, in the following diagram for $x = 31524 \in S_5$ we observe that the number of points north-west of the point $(4, 2)$ is 4 , which corresponds to the number of points in the shaded area, thus $x[4, 2] = 3$.

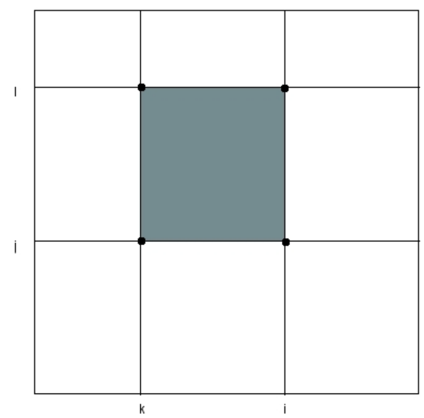


Remark 1.5. Note that for any $x \in S_n$ $x[n, i] = n + 1 - i$ and $x[i, 1] = i$.

The following lemma is essentially to determining conditions for Bruhat comparability.

Lemma 1.6. $x[i, j] - x[k, j] - x[i, l] + x[k, l] = |\{a \in [k + 1, i] : j \leq x(a) < l\}|$ $\forall 1 \leq k \leq i \leq n$ *and* $\forall 1 \leq j \leq l \leq n$.

The proof is evident by considering the diagram below.

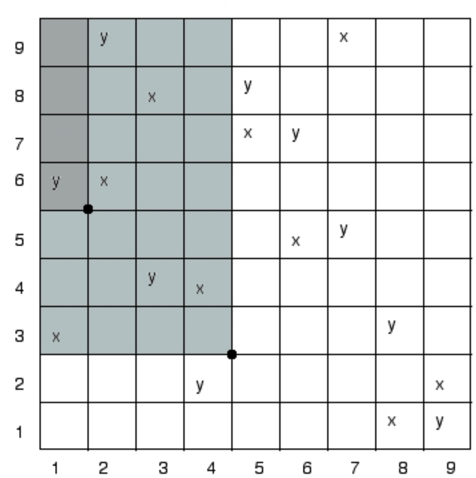


We can now state our theorem characterizing Bruhat comparability.

Theorem 1.7. *Let $x, y \in S_n$, then the following are equivalent:*

(i) $x \leq y$
(ii) $x[i,j] \leq y[i,j] \ \forall i, j \in [n]$

As an application, let us determine if $x = 368475912$ and $y = 694287531$ are comparable in Bruhat order. Considering the corresponding overlapped diagrams we observe (see the figure below) that $x[1,6] = 0 < 1 = y[1,6]$, and $x[4,3] = 4 > 3 = y[4,3]$. Consequently, x and y are not comparable in Bruhat order.



Proof. (i) $\Rightarrow$ (ii). Suppose that $x \leq y$, and without loss of generality assume $x \to y$. Then there exist a and b, $1 < a < b < n$, such that $y = x(a,b)$ and $x(a) < x(b)$. By definition (1.1) this implies that

$$y[i,j] = \begin{array}{ll} x[i,j]+1 & , \quad \text{if } a \leq i < b, \quad x(a) < j \leq x(b) \\ x[i,j] & , \quad \text{otherwise} \end{array}$$

so (ii) follows.
(i) $\Leftarrow$ (ii). see Björner and Brenti.

$\square$