Homework 1

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Problem 1. Show that the dominance partial order on partitions of n satisfies

$$\lambda \trianglelefteq \mu \quad \Longleftrightarrow \quad \lambda^t \trianglerighteq \mu^t,$$

where the t denotes the transpose of the partition.

Problem 2. For $1 \leq i < j \leq n$, define the raising operator R_{ij} on $\mathbb{Z}^n$ by

$$R_{ij}(\nu_1, \ldots, \nu_n) = (\nu_1, \ldots, \nu_i + 1, \ldots, \nu_j - 1, \ldots, \nu_n).$$

(1) Show that the dominance order $\trianglelefteq$ is the transitive closure of the relation on partitions $\lambda \to \mu$ if $\mu = R_{ij}\lambda$ for some $i < j$.
(2) Show that μ covers λ if and only if $\mu = R_{ij}\lambda$, where i, j satisfy the following condition: either $j = i + 1$ or $\lambda_i = \lambda_j$ (or both).
(3) Find the smallest n such that the dominance order on partitions of n is not a total ordering, and draw its Hasse diagram.

Problem 3. Let h_i be the complete homogeneous symmetric functions. Show that $u_i \in \Lambda$ satisfying $u_0 = 1$ and

$$\sum_{i=0}^{n} (-1)^i u_i h_{n-i} = 0 \qquad \text{for all } n \geq 1$$

are uniquely determined.

Problem 4. Let $f \in \Lambda^n$, and for any $g \in \Lambda^n$ define $g_k \in \Lambda^{nk}$ by

$$g_k(x_1, x_2, \ldots) = g(x_1^k, x_2^k, \ldots).$$

Show that

$$\omega f_k = (-1)^{n(k-1)}(\omega f)_k.$$

Problem 5. Let $w \in S_n$ be an element of the symmetric group of cycle type λ. Give a direct bijective proof that the number of elements $v \in S_n$ commuting with w is equal to

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$$

where $m_i = m_i(\lambda)$ is the number of parts of λ of size i.

Problem 6. Show that

$$\prod_{\lambda \vdash n} \prod_{i \geq 1} m_i(\lambda)! = \prod_{\lambda \vdash n} \prod_{i \geq 1} i^{m_i(\lambda)}.$$

Problem 7. The symmetric functions $f_\lambda = \omega m_\lambda$ are sometimes called the "forgotten" symmetric functions. Show that the matrix of coefficients of the forgotten symmetric functions f_λ expressed in terms of monomial symmetric functions m_λ is the transpose of the matrix of the elementary functions e_λ expressed in terms of the complete homogeneous symmetric functions h_λ.

Problem 8. Let ∂p_k be the operator on symmetric functions given by partial differentiation with respect to p_k, under the identification of symmetric functions with polynomials $f \in \mathbb{Q}[p_1, p_2, \ldots]$. Show that ∂p_k is adjoint with respect to the scalar product to the operator of multiplication by p_k/k.