Homework 3

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Problem 1. Show that
$$\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i+y_j)=\sum_{\lambda} s_\lambda(x)s_{\tilde{\lambda}^t}(y)$$
where the sum is over all partitions λ with $\ell(\lambda)\le m$ and $\lambda_1\le n$ and $\tilde{\lambda}$ is defined as
$$\tilde{\lambda}=(n-\lambda_m, n-\lambda_{m-1},\ldots,n-\lambda_1).$$

Problem 2. Let $D_\mu:\Lambda\to\Lambda$ be the linear transformation given by
$$D_\mu(s_\lambda)=s_{\lambda/\mu}.$$
Show that $D_\mu D_\lambda = D_\lambda D_\mu$.

Problem 3. If R is a ring, then an additive group homomorphism $D: R\to R$ is called a derivation if $D(fg)=(Df)g+f(Dg)$ for all $f,g\in R$.

(1) Show that the linear transformation $\Lambda\to\Lambda$ defined by $D(s_\lambda)=s_{\lambda/1}$ is a derivation.
(2) Show that the bilinear operation $[f,g]$ on Λ given by
$$[s_\lambda,s_\mu]=s_{\lambda/1}s_\mu - s_\lambda s_{\mu/1}$$
defines a Lie algebra structure on Λ. That is, the Jacobi identity holds
$$[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0.$$

Problem 4. If A is an algebra over a field k, then $A\otimes_k A$ is an algebra with multiplication characterized uniquely by $(a\otimes b)(c\otimes d)=ac\otimes bd$. A coproduct is an algebra homomorphism $\Delta: A\to A\otimes A$ which is coassociative in the sense that the two maps
$$(1\otimes\Delta)\circ\Delta \quad\text{and}\quad (\Delta\otimes 1)\circ\Delta$$
from A to $A\otimes A\otimes A$ are equal. (This axiom is dual to the associative law for multiplication $\mu: A\otimes A\to A$, which can be formulated as $\mu\circ(1\otimes\mu)=\mu\circ(\mu\otimes 1)$.) Here $1: A\to A$ is the identity map.

Taking Λ to be the algebra of symmetric functions with coefficients in $k=\mathbb{Q}$ and identifying $\Lambda\otimes\Lambda$ with $\Lambda(X)\Lambda(Y)$, show that $\Delta(f)=f(X,Y)$ defines a coproduct on Λ.

Remark: An algebra equipped with a coproduct is called a bialgebra. If we also define the counit $\epsilon : \Lambda \to \mathbb{Q}$ by $\epsilon(f) = \langle f, 1 \rangle = f(0, 0, \ldots)$ and the antipode $S : \Lambda \to \Lambda$ by $Sf = f(-X)$, these together with Δ can be shown to satisfy the axioms of a Hopf algebra.

Problem 5. The relevance of this problem will become clear when we talk about differential posets in class.

(1) Let $U : \Lambda \to \Lambda$ and $D : \Lambda \to \Lambda$ be linear transformations defined by

$$U(f) = p_1 f$$
$$D(f) = \frac{\partial}{\partial p_1} f$$

where $\partial/\partial p_1$ is applied to f written as a polynomial in the p_i's. Show that

$$DU - UD = I$$
$$DU^k = kU^{k-1} + U^k D$$

where I is the identity operator.

(2) Now let us work in Young's lattice Y and define

$$D(\lambda) = \sum_{\lambda^- \lhd \lambda} \lambda^-$$
$$U(\lambda) = \sum_{\lambda^+ \rhd \lambda} \lambda^+$$

where the sum over λ^- is over all partitions begin covered by λ in Y and the sum over λ^+ is over all partitions covering λ in Y. Show that

$$DU - UD = I$$
$$DU^k = kU^{k-1} + U^k D.$$

(3) Derive from (2) that

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$