Homework 5

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Problem 1. Let $|\lambda| = |\mu| = n$. Show that $\langle h_\lambda, h_\mu \rangle$ is equal to the number of double cosets $S_\lambda w S_\mu$ in the symmetric group S_n, where $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell}$, embedded as a subgroup of S_n, similarly for S_μ, and $w \in S_n$.

Problem 2. Define the Kronecker product on symmetric functions in terms of the power-sum basis by

$$p_\lambda \star p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda.$$

Equivalently, the symmetric functions p_λ / z_λ are orthogonal idempotents with respect to $\star$.

(1) Prove that the Kronecker coefficients $a_{\lambda\mu\nu}$ defined by

$$s_\mu \star s_\nu = \sum_\lambda a_{\lambda\mu\nu} s_\lambda$$

are invariant under permuting the indices λ, μ, ν.

(2) Show that if $f \in \Lambda^n$, then $e_n \star f = wf$.

Remark: In fact $a_{\lambda\mu\nu}$ are non-negative integers. It is an open problem to find a combinatorial rule for the computation of the Kronecker coefficients, except for some special cases.