Homework 7

due March 7, 2014 in class

1. Prove the following identities in an arbitrary ring R from the axioms.

(a) $0a = 0$

(b) $-a = (-1)a$

(c) $(-a)b = -(ab)$

2. (Artin 11.1.1) Prove that $7 + \sqrt[3]{2}$ and $\sqrt{3} + \sqrt{-5}$ are algebraic numbers.

3. (Artin 11.1.3) Let $\mathbb{Q}[\alpha, \beta]$ denote the smallest subring of $\mathbb{C}$ containing $\mathbb{Q}, \alpha = \sqrt{2}$, and $\beta = \sqrt{3}$, and let $\gamma = \alpha + \beta$. Prove that $\mathbb{Q}[\alpha, \beta] = \mathbb{Q}[\gamma]$. Is $\mathbb{Z}[\alpha, \beta] = \mathbb{Z}[\gamma]$?

4. (similar to Artin 11.1.6) In each case, decide whether or not S is a subring of R.

(a) S is the set of all rational numbers of the form a/b, where b is not divisible by 3, and $R = \mathbb{Q}$.

(b) S is the set of all real matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, and R is the set of all 2×2 matrices.

5. (similar to Artin 11.1.7) In each case, decide whether the given structure forms a ring. If it is not a ring, determine which of the ring axioms hold and which fail:

(a) U is an arbitrary set, and R is the set of subsets of U. Addition and multiplication of elements of R are defined by the rules $A + B = A \cup B$ and $A \cdot B = A \cap B$ (note that in Artin 11.1.7 there is an additional $-(A \cap B)$ in the sum).

(b) R is the set of continuous functions $\mathbb{R} \to \mathbb{R}$. Addition and multiplication are defined by the rules $[f + g](x) = f(x) + g(x)$ and $[f \circ g](x) = f(g(x))$.

6. Determine all rings which contain the zero ring as a subring.

7. (Artin 11.1.8) Describe the group of units in each ring. (The operation in the group of units is multiplication from the ring).

(a) $\mathbb{Z}/12\mathbb{Z}$.

(b) $\mathbb{Z}/8\mathbb{Z}$.

(c) $\mathbb{Z}/n\mathbb{Z}$.