1 Defining Representation of S_n

For convenience, I will write all column vectors as row vectors.

For the defining representation X of S_n over $\mathbb{C}$ with the standard basis, the span of $\mu = (1, 1, 1..., 1)$ is clearly G-invariant (that is, for all $v \in Span(\mu)$, $\sigma v = v \ \forall \ \sigma \in S_n$); we will call this subrepresentation X^{Triv} on subspace U (I leave it to the reader to show that this is the trivial representation). Then X may be written as $X^{Triv} \oplus X^{\perp}$ (this is proven for finite dimensional representations in 150C). It follows that the orthogonal complement V of U ($X^{\perp}$ is over V) is also G-invariant; were it not, there would exist some $v \in V$ and some $X_g \in X$ such that $X_g v \in U \Rightarrow X_{g^{-1}} X_g v \in V \Rightarrow \ \exists u \in U$ such that $X_{g^{-1}} u \in X^{\perp} \lightning$. Note V is of dimension $n - 1$.

Proposition 1.1. *$X^{\perp}$ is an irreducible representation.*

Proof. It will prove useful to know what vectors are contained in V. I claim it is the set $\{v \in \mathbb{C}^n | \Sigma_{i=1}^n v_i = 0\}$. This is not difficult to see, as $\langle v, l\mu \rangle = l\Sigma_{i=1}^n v_i$ (where $\langle \ , \ \rangle$ is the standard dot product) and for orthogonal vectors this product must be 0.

Now, for the sake of contradiction, assume $X^{\perp}$ has some subrepresentation. This is equivalent to stating there is some non-trivial G-invariant subspace of V, which we will call W. Let $\xi \in W$; as $W \subset V$, ξ may be expressed as a sum of basis vectors of V. Assume ξ has only two non-zero terms, in which case one is the negative of the other, say γ and $-\gamma$. Multiply ξ by $\frac{1}{|\gamma|}$ and we will have a vector, call it ξ', that has only 0's, one 1 and one -1 as entries. We can use ξ' to define a basis of V; take the set $\{X_{(ij)}\xi' \mid \text{i is the index of the negative entry of } \xi' \text{ and } 1 \leq j \leq n, i \neq j\}$ (if it was unclear, $X_{(ij)}$ refers to the representation of the cycle (ij)). This set has $n - 1$ linearly independent vectors and is thus a basis. However, because W is G-invariant, every vector in that set is also in W, so a complete basis for V is contained in W which implies $dim(W) = dim(V) \lightning$.

Now assume ξ has more than 2 non-zero terms. Let $j - 1$ equal the index of the first non-zero entry of ξ and let q be a vector such that $q_{j-1} = 1$, $q_j = -1$ and 0 elsewhere; take the basis of V, $\mathcal{B} = \{X_{(ji)}q \mid 1 \leq i \leq n, i \neq j\}$. Let $q_1 = q$, $q_2 = X_{(j \ j+1)}q$, $q_3 = X_{(j \ j+2)}q$ and so on. Then we have

$$\xi = m_1 q_1 + m_2 q_2 + ...$$

Let $m_k q_k$ be the first term such that $m_k \neq 0$. Then we have

$$\xi = m_k q_k + m_{k+1} q_{k+1} + ...$$

Let

$$\begin{aligned}
\xi' &= X_{(j-1\ \ k)}\xi = X_{(j-1\ \ k)} m_k q_k + X_{(j-1\ \ k)} m_{k+1} q_{k+1} + ... \\
&= -m_k q_k + X_{(j-1\ \ k)} m_{k+1} q_{k+1} + ... \\
&\Rightarrow \xi + \xi' = X_{(j-1\ \ k)} m_{k+1} q_{k+1} + X_{(j-1\ \ k)} m_{k+1} q_{k+1} + ...
\end{aligned}$$

Call the resulting vector ξ''. Notice that the k^{th} entry of ξ'' is now 0 and that all entries with index less than $j-1$ remain 0, so we have reduced the number of non-zero terms by at least 1. The previous method used in class did not account for the situation in which $\xi'' = 0$, but this does not occur here. $X_{(j-1\ \ k)}\xi$ only differs from ξ in the entries with index $j-1$ and k, so all other entries will be doubled in ξ'' and by assumption there are at least 3 non-zero entries. Furthermore, this method will always leave at least 2 non-zero entries in ξ'. To see this assume there is only one non-zero entry. However, that would imply the sum of the entries cannot be 0, in which case V would not be a G-invariant subspace $\lightning$. Thus we may repeat the process until there are exactly 2 non-zero entries and then proceed as above.

$\square$