Problem 1. Let $C(x, y) = \prod_{i,j \geq 1} (1 - x_i y_j)^{-1}$. Show that then for all $f \in \Lambda$

$$\langle C(x, y), f(x) \rangle = f(y)$$

where the scalar product is taken in the $x$ variables. In other words, $C(x, y)$ is a reproducing kernel for the scalar product.

Problem 2. Show that

$$\prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j) = \sum_{\lambda} s_{\lambda}(x)s_{\bar{\lambda}}(y)$$

where the sum is over all partitions $\lambda$ with $\ell(\lambda) \leq m$ and $\lambda_1 \leq n$ and $\bar{\lambda}$ is defined as

$$\bar{\lambda} = (n - \lambda_m, n - \lambda_{m-1}, \ldots, n - \lambda_1).$$

Problem 3. Let $D_\mu : \Lambda \rightarrow \Lambda$ be the linear transformation given by

$$D_\mu(s_{\lambda}) = s_{\lambda/\mu}.$$ 

Show that $D_\mu D_\lambda = D_\lambda D_\mu$.

Problem 4. If $R$ is a ring, then an additive group homomorphism $D : R \rightarrow R$ is called a derivation if $D(fg) = (Df)g + f(Dg)$ for all $f, g \in R$.

1. Show that the linear transformation $\Lambda \rightarrow \Lambda$ defined by $D(s_{\lambda}) = s_{\lambda/1}$ is a derivation.

2. Show that the bilinear operation $[f, g]$ on $\Lambda$ given by

$$[s_{\lambda}, s_{\mu}] = s_{\lambda/1}s_{\mu} - s_{\lambda}s_{\mu/1}$$

defines a Lie algebra structure on $\Lambda$. That is, the Jacobi identity holds

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0.$$ 

Problem 5. If $A$ is an algebra over a field $k$, then $A \otimes_k A$ is an algebra with multiplication characterized uniquely by $(a \otimes b)(c \otimes d) = ac \otimes bd$. A coproduct is an algebra homomorphism $\Delta : A \rightarrow A \otimes A$ which is coassociative in the sense that the two maps

$$(1 \otimes \Delta) \circ \Delta \quad \text{and} \quad (\Delta \otimes 1) \circ \Delta$$
from $A$ to $A \otimes A \otimes A$ are equal. (This axiom is dual to the associative law for multiplication $\mu : A \otimes A \to A$, which can be formulated as $\mu \circ (1 \otimes \mu) = \mu \circ (\mu \otimes 1)$.) Here $1 : A \to A$ is the identity map.

Taking $\Lambda$ to be the algebra of symmetric functions with coefficients in $k = \mathbb{Q}$ and identifying $\Lambda \otimes \Lambda$ with $\Lambda(\mathbb{X})\Lambda(\mathbb{Y})$, show that $\Delta(f) = f(X,Y)$ defines a coproduct on $\Lambda$.

**Remark:** An algebra equipped with a coproduct is called a bialgebra. If we also define the counit $\epsilon : \Lambda \to \mathbb{Q}$ by $\epsilon(f) = \langle f, 1 \rangle = f(0,0,\ldots)$ and the antipode $S : \Lambda \to \Lambda$ by $Sf = f(-X)$, these together with $\Delta$ can be shown to satisfy the axioms of a Hopf algebra.

**Problem 6.** The relevance of this problem will become clear when we talk about differential posets in class.

1. Let $U : \Lambda \to \Lambda$ and $D : \Lambda \to \Lambda$ be linear transformations defined by

   $$U(f) = p_1 f$$
   $$D(f) = \frac{\partial}{\partial p_1} f$$

   where $\partial/\partial p_1$ is applied to $f$ written as a polynomial in the $p_i$'s. Show that

   $$DU - UD = I$$
   $$DU^k = kU^{k-1} + U^k D$$

   where $I$ is the identity operator.

2. Now let us work in Young's lattice $Y$ and define

   $$D(\lambda) = \sum_{\lambda^- < \lambda} \lambda^-$$
   $$U(\lambda) = \sum_{\lambda^+ \triangleright \lambda} \lambda^+$$

   where the sum over $\lambda^-$ is over all partitions being covered by $\lambda$ in $Y$ and the sum over $\lambda^+$ is over all partitions covering $\lambda$ in $Y$. Show that

   $$DU - UD = I$$
   $$DU^k = kU^{k-1} + U^k D.$$ 

3. Derive from (2) that

   $$\sum_{\lambda^n \sim n} (f^\lambda)^2 = n!$$