Homework 5
due March 17, 2017

Problem 1. Let $|\lambda| = |\mu| = n$. Show that $\langle h_\lambda, h_\mu \rangle$ is equal to the number of double cosets $S_\lambda w S_\mu$ in the symmetric group $S_n$, where $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell}$, embedded as a subgroup of $S_n$, similarly for $S_\mu$, and $w \in S_n$.

Problem 2. Define the Kronecker product on symmetric functions in terms of the power-sum basis by

$$p_\lambda \ast p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda.$$ 

Equivalently, the symmetric functions $p_\lambda/z_\lambda$ are orthogonal idempotents with respect to $\ast$.

1. Prove that the Kronecker coefficients $a_{\lambda\mu\nu}$ defined by

$$s_\mu \ast s_\nu = \sum \lambda a_{\lambda\mu\nu} s_\lambda$$

are invariant under permuting the indices $\lambda, \mu, \nu$.

2. Show that if $f \in \Lambda^n$, then $e_n \ast f = w f$.

Remark: In fact $a_{\lambda\mu\nu}$ are non-negative integers. It is an open problem to find a combinatorial rule for the computation of the Kronecker coefficients, except for some special cases.

Problem 3. The principle specialization of a symmetric function in the variables $\{x_1, x_2, \ldots, x_m\}$ is obtained by replacing $x_i$ by $q^i$ for all $i$.

(a) Show that the Schur function specialization $s_\lambda(q, q^2, \ldots, q^m)$ is the generating function for semistandard $\lambda$-tableaux with all entries of size at most $m$.

(b) Define the content of cell $(i, j)$ to be $c_{i,j} = j - i$. Prove that

$$s_\lambda(q, q^2, \ldots, q^m) = q^{m(\lambda)} \prod_{(i,j) \in \lambda} \frac{1 - q^{c_{i,j}+m}}{1 - q^{h_{i,j}}}$$

where $m(\lambda) = \sum_{i \geq 1} i \lambda_i$ and $h_{i,j}$ is the hook length of the cell $(i, j)$ in $\lambda$. 
Problem 4. Let \( r \) be a positive integer. A poset \( A \) is \( r \)-differential if it satisfies the definition from class with the second condition replaced by

- If \( a \in A \) covers \( k \) elements for some \( k \), then it is covered by \( k + r \) elements.

Prove the following statements about \( r \)-differential posets \( A \).

(a) The rank cardinalities \( |A_n| \) are finite for all \( n \geq 0 \). (This implies that the operations \( D \) and \( U \) are well-defined).
(b) Let \( A \) be a graded poset with \( A_n \) finite for all \( n \geq 0 \). Then \( A \) is \( r \)-differential if and only if \( DU - UD = rI \).
(c) In any \( r \)-differential poset

\[
\sum_{a \in A_n} (f^a)^2 = r^n n!,
\]

where \( f^a \) is the number of saturated \( \emptyset - a \) chains.
(d) If \( A \) is \( r \)-differential and \( B \) is \( s \)-differential, then the product \( A \times B \) is \( (r + s) \)-differential. So if \( A \) is 1-differential, then the \( r \)-fold product \( A^r \) is \( r \)-differential.

Problem 5. Show that the crystal operators \( f_i \) and \( e_i \) respect the Knuth relations, that is, if \( w \overset{K}{\succeq} v \), then \( e_i w \overset{K}{\preceq} e_i v \) (resp. \( f_i w \overset{K}{\succeq} f_i v \)) as long as \( e_i \) (resp. \( f_i \)) does not annihilate \( w \). Furthermore, \( w \) and \( f_i w \) have the same recording tableau under Schensted insertion. This proves in particular, that the crystal operators can be defined on semistandard tableaux.