Homework 1
due January 18

Problem 1. Show that the dominance partial order on partitions of \( n \) satisfies
\[
\lambda \trianglelefteq \mu \iff \lambda^t \succeq \mu^t,
\]
where the \( t \) denotes the transpose of the partition.

Problem 2. For \( 1 \leq i < j \leq n \), define the raising operator \( R_{ij} \) on \( \mathbb{Z}^n \) by
\[
R_{ij}(\nu_1, \ldots, \nu_n) = (\nu_1, \ldots, \nu_i + 1, \ldots, \nu_j - 1, \ldots, \nu_n).
\]

(1) Show that the dominance order \( \trianglelefteq \) is the transitive closure of the relation on partitions \( \lambda \rightarrow \mu \) if \( \mu = R_{ij}\lambda \) for some \( i < j \).
(2) Show that \( \mu \) covers \( \lambda \) if and only if \( \mu = R_{ij}\lambda \), where \( i, j \) satisfy the following condition: either \( j = i + 1 \) or \( \lambda_i = \lambda_j \) (or both).
(3) Find the smallest \( n \) such that the dominance order on partitions of \( n \) is not a total ordering, and draw its Hasse diagram.

Problem 3. Let \( h_i \) be the complete homogeneous symmetric functions. Show that \( u_i \in \Lambda \) satisfying \( u_0 = 1 \) and
\[
\sum_{i=0}^{n} (-1)^i u_i h_{n-i} = 0 \quad \text{for all } n \geq 1
\]
are uniquely determined.

Problem 4. Let \( w \in S_n \) be an element of the symmetric group of cycle type \( \lambda \). Give a direct bijective proof that the number of elements \( v \in S_n \) commuting with \( w \) is equal to
\[
z_{\lambda} = 1^{m_1} m_1! 2^{m_2} m_2! \cdots
\]
where \( m_i = m_i(\lambda) \) is the number of parts of \( \lambda \) of size \( i \).

Problem 5. Show that
\[
\prod_{\lambda \vdash n} \prod_{i \geq 1} m_i(\lambda)! = \prod_{\lambda \vdash n} \prod_{i \geq 1} i^{m_i(\lambda)}.
\]