Homework 2

due February 1

Problem 1. Let $f \in \Lambda^n$, and for any $g \in \Lambda^n$ define $g_k \in \Lambda^{nk}$ by

$$g_k(x_1, x_2, \ldots) = g(x_1^k, x_2^k, \ldots).$$

Show that

$$\omega f_k = (-1)^{n(k-1)}(\omega f)_k.$$

Problem 2. The symmetric functions $f_\lambda = \omega m_\lambda$ are sometimes called the "forgotten" symmetric functions. Show that the matrix of coefficients of the forgotten symmetric functions f_λ expressed in terms of monomial symmetric functions m_λ is the transpose of the matrix of the elementary functions e_λ expressed in terms of the complete homogeneous symmetric functions h_λ.

Problem 3. Let ∂p_k be the operator on symmetric functions given by partial differentiation with respect to p_k, under the identification of symmetric functions with polynomials $f \in \mathbb{Q}[p_1, p_2, \ldots]$. Show that ∂p_k is adjoint with respect to the scalar product to the operator of multiplication by p_k/k.

Problem 4. Using the symmetry of the RSK algorithm, show the following:

(1) A permutation π is an involution if and only if $P(\pi) = Q(\pi)$, where $(P(\pi), Q(\pi))$ correspond to π under the RSK algorithm.
(2) The number of involutions of S_n is $\sum_{\lambda \vdash n} f^\lambda$.
(3) The number of fixed points in an involution π is the number of columns of odd length in $P(\pi)$.
(4) We have

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \sum_{\substack{\lambda \vdash 2n \\ \lambda^t \text{ even}}} f^\lambda,$$

where λ^t even means that every part in λ^t is even.
(5) There is a bijection $M \longleftrightarrow T$ between symmetric $\mathbb{N}$-matrices of finite support and semistandard Young tableaux such that the trace of M is the number of columns of odd length of T.

(6) The following equations hold

$$\sum_{\lambda} s_\lambda = \prod_i \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j}$$

$$\sum_{\lambda^t \text{ even}} s_\lambda = \prod_{i<j} \frac{1}{1-x_i x_j}.$$

Problem 5. Suppose $\pi = x_1 x_2 \dots x_n \in S_n$ is a permutation in one-line notation such that $P = P(\pi)$ has rectangular shape. Let the complement of π be

$$\pi^c = y_1 y_2 \dots y_n$$

where $y_i = n+1-x_i$ for all i. Also define the complement of a rectangular standard tableau P with n entries to be the array obtained by replacing P_{ij} with $n+1-P_{ij}$ for all (i,j) and then rotating the result by $180°$. Show that

$$P(\pi^c) = (P^c)^t.$$

Problem 6. For any symmetric polynomial f, let $f^\perp$ be the operator adjoint to multiplication by f with respect to the Hall inner product, that is, $\langle f^\perp g, h\rangle = \langle g, fh\rangle$ for all $g, h \in \Lambda$.

(1) Find a formula for $h_k^\perp m_\lambda$, expressed again in terms of monomial symmetric functions m_μ.
(2) Show that the basis of monomial symmetric functions is uniquely characterized by the formula from the previous part.

Problem 7. A plane partition of n is a sequence of ordinary partitions $\lambda = (\lambda^{(1)} \supseteq \cdots \supseteq \lambda^{(k)})$ of total size $\sum_i |\lambda^{(i)}| = n$, weakly decreasing in the sense that the diagram of each $\lambda^{(i)}$ is contained in that of the preceding one. The diagram of a plane partition is the three-dimensional array in $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ whose i-th horizontal layer is the diagram of $\lambda^{(i)}$.

Find a bijection between plane partitions whose diagram fits inside a $k \times \ell \times m$ box and semi-standard Young tableaux of shape (k^ℓ) with entries in $[\ell + m]$.