Crystal skeletons: Combinatorics and axioms

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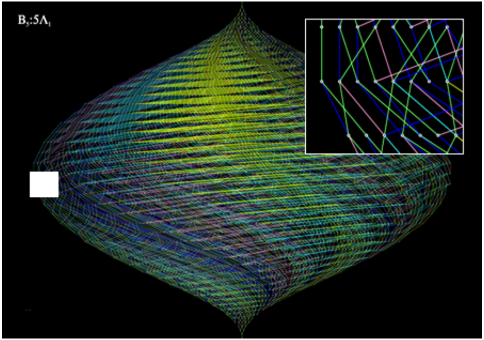

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which originated in statistical mechanics and quantum groups
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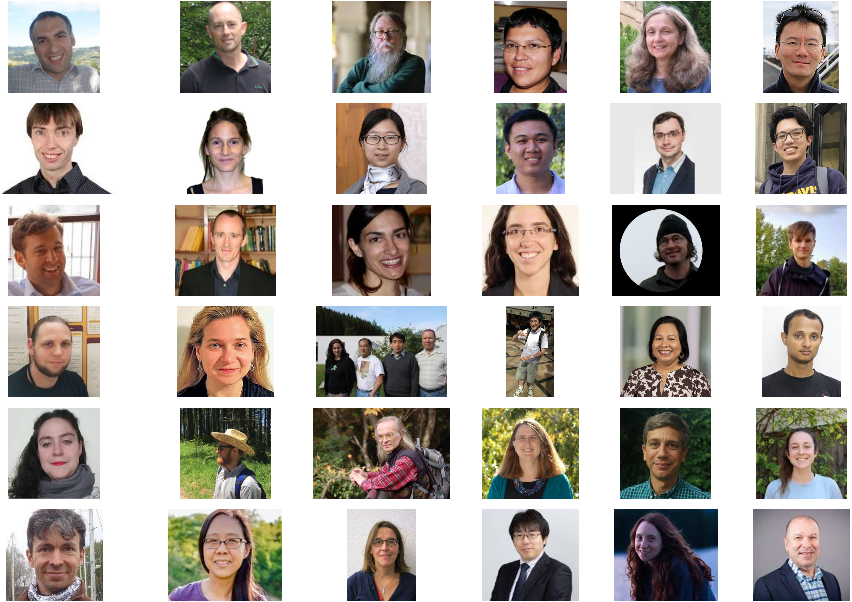
Many crystal collaborators over the years:
Assaf, Bandlow, Benkart, Brauner, Bump, Colmenarejo, Corteel, Daugherty, Deka, Fourier, Gillespie, Harris, Hawkes, Hersh, Jones, Kirillov, Lam, Lenart, Mason, Morse, Naito, Okado, Orellana, Pan, Panova, Pappe, Paramonov, Paul, Pfannerer, Poh, Sagaki, Sakamoto, Saliola, Scrimshaw, Shimozono, Simone, Sternberg, Thiéry, Tingley, Wang, Warnaar, Yip, Zabrocki



Outline

1. Origins
2. Crystals
3. Applications
4. Quasicrystals
5. Crystal skeletons
6. Quasicrystal skeletons

Lie algebras

Lie algebra $\mathfrak{sl}_2$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Relations

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = h$$

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Weight space decomposition

$$V = \bigoplus_{\lambda} V(\lambda) \quad \text{where} \quad V(\lambda) = \{v \in V \mid hv = \lambda v\}$$

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$$eV(\lambda) \subset V(\lambda + 2) \qquad fV(\lambda) \subset V(\lambda - 2)$$

Quantum groups

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Representations

$(m+1)$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-representation

$$V_{(m)} = \{u, f^{(1)}u, \ldots, f^{(m)}u\}$$

$$\text{where} \quad eu = 0 \quad Ku = q^m u \quad f^{(k)}u = \frac{1}{[k]_q!} f^k u \quad [k]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$$

Motivation for crystal bases

2-dimensional $U_q(\mathfrak{sl}_2)$-representation $V_{(1)}$

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$$\mathrm{u} \underset{e}{\overset{f}{\rightleftarrows}} \mathrm{v}$$

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$$V_{(1)} \otimes V_{(1)} \cong V_{(2)} \oplus V_{(0)} \qquad \begin{aligned} V_{(2)} &= \{u \otimes u, u \otimes v + qv \otimes u, v \otimes v\} \\ V_{(0)} &= \{v \otimes u - qu \otimes v\} \end{aligned}$$

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Crystal basis

Pick leading term $(q \to 0)$

$$B_{(1)} \otimes B_{(1)} \cong B_{(2)} \oplus B_{(0)} \qquad B_{(2)} = \{u \otimes u, u \otimes v, v \otimes v\}$$

$$B_{(0)} = \{v \otimes u\}$$

Motivation for crystal bases

$u = \boxed{1} \qquad v = \boxed{2}$

$$B_{\square}$$

$$\begin{tikzcd} \boxed{1} \arrow[d, "1"] \\ \boxed{2} \end{tikzcd}$$

$$B_{\square} \otimes B_{\square}$$

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Reason 1

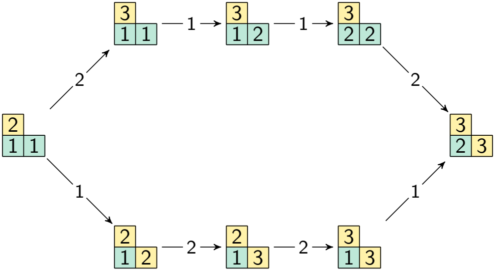
Crystal bases provide combinatorial models of representation theory.

Outline

Crystals

Combinatorial representation theory of $\mathfrak{sl}_n$

Crystal: $B(2,1)$



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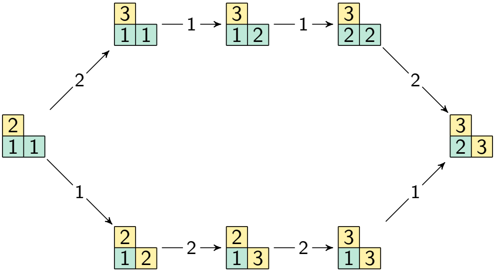
Reason 2

Crystal bases can be described by "easy" combinatorial rules.

Crystals

Character of crystal $\mathrm{ch}B(\lambda) = s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(T)}$

Crystal: $B(2,1)$



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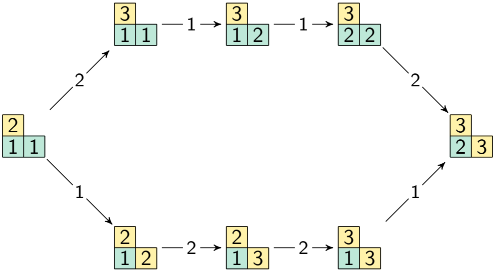

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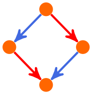
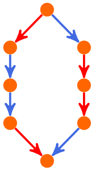
Reason 3

Characters of crystals are (sums of) Schur functions.

Local characterization

Local characterization of simply-laced crystals associated to representations (Stembridge 2003)





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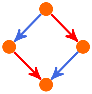
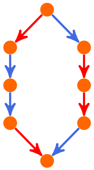
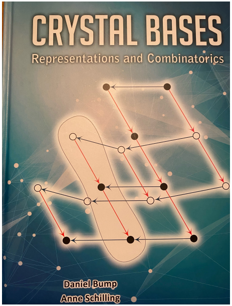
Combinatorial theory of crystals without quantum groups:







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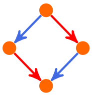

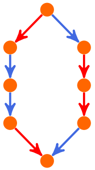

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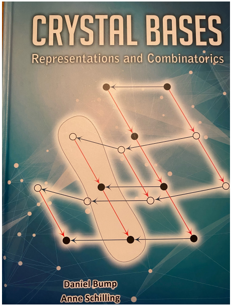

Reason 4

Crystal graphs can be characterized by local combinatorial rules.

Outline

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$$s_\lambda = \sum_{\alpha \in S_\ell \cdot \lambda} \mathcal{YQS}_\alpha \quad \text{and} \quad \mathcal{YQS}_\alpha = \sum_\beta d_{\alpha,\beta} F_\beta$$

- $\mathcal{YQS}_\alpha$ Young quasisymmetric Schur function indexed by composition α

Motivation

Tiling and contraction

$$\begin{array}{ccccc}
\text{Crystal } B(\lambda) & & \text{Crystal skeleton } \mathrm{CS}(\lambda) & & \\
\mathrm{char}B(\lambda) = s_\lambda & & \mathrm{char}\mathrm{CS}(\lambda) = s_\lambda & & \\
\rotatebox{90}{$\subseteq$} & \longrightarrow & \rotatebox{90}{$\subseteq$} & \longrightarrow & \text{Bruhat order} \\
\text{Quasicrystal } Q_\beta & & \text{Quasicrystal skeleton } \mathrm{QCS}_\alpha & & \\
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\end{array}$$

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Reason 5

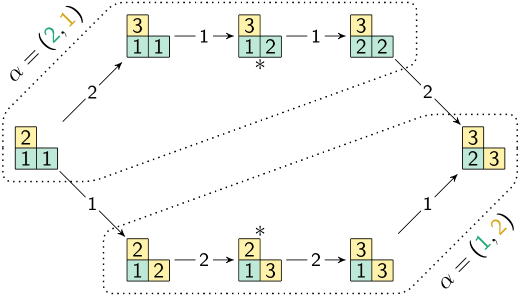
Crystals can be used to prove positive Schur expansion.

Outline

Quasicrystals

Group tableaux by standardization – labeled by descent composition

Crystal: $B(2,1)$



Quasicrystals

2017 $\rightarrow$ Cain, Gray, Guilherme, Malheiro, Ribeiro, Rodrigues, Rodrigues
quasicrystals and hypoplactic monoids

2023 Maas-Gariépy

- Quasicrystal: Q_T indexed by standard Young tableaux $T \in \mathrm{SYT}(\lambda)$
- Vertices:

$$Q_T = \{b \in B(\lambda) \mid \mathrm{std}(b) = T\}$$

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- Vertices:
$$Q_T = \{b \in B(\lambda) \mid \mathsf{std}(b) = T\}$$
- Edges: $f_i(T) = T'$ in crystal is quasicrystal edge
if and only if no $i+1$ is paired with i

Quasicrystals

Theorem

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SYT(λ) set of standard Young tableaux of shape λ
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Definition

i is a descent in T if $i+1$ is in higher row of the tableau
$d_1 < d_2 < \cdots < d_k$ descents in T

$$\mathrm{Des}(T) = (d_1, d_2 - d_1, \ldots, d_k - d_{k-1}, n - d_k)$$

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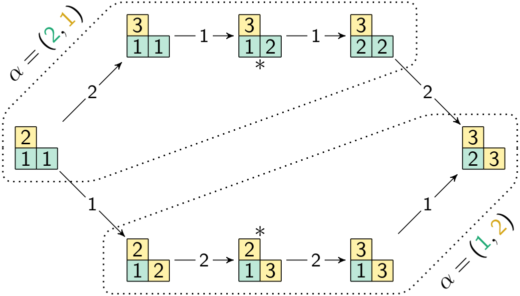
Example

$$T = \begin{array}{|c|c|c|c|} \hline 7 \\ \cline{1-1} 3 & 5 \\ \cline{1-2} 1 & 2 & 4 & 6 \\ \hline \end{array} \qquad \text{descents } \{2,4,6\} \quad \mathrm{Des}(T) = (2,2,2,1)$$

Quasicrystals

Character of quasicrystal: $\mathrm{ch}\,Q_T = F_{\mathrm{Des}(T)}$

Crystal: $B(2,1)$



Schur function $s_{(2,1)} = F_{(2,1)} + F_{(1,2)}$

Gessel's quasisymmetric functions

Gessel's quasisymmetric functions (1984)

$$F_\alpha = \sum_{\substack{\beta \preccurlyeq \alpha \\ \text{refinement}}} M_\beta \qquad \text{with} \qquad M_\beta = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$$

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Example

$$\begin{aligned} F_{(1,2,1)} =& M_{(1,2,1)} + M_{(1,1,1,1)} \\ =& x_1 x_2^2 x_3 + x_1 x_2^2 x_4 + x_2 x_3^2 x_4 + \cdots \\ &+ x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5 + \cdots \end{aligned}$$

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Reason 6

Characters of quasicrystals give Gessel's quasisymmetric function.

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Tiling and contraction

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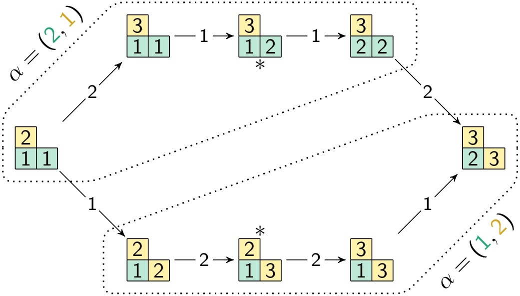
Based on joint work with

Sarah Brauner, Sylvie Corteel, and Zajj Daugherty
preprint arXiv:2503.14782

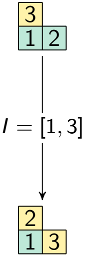
Crystal skeletons

2023 Maas-Gariépy: Contract quasicrystals to a point

Crystal: $B(2,1)$



Crystal skeleton: $\mathrm{CS}(2,1)$



Schur function $s_{(2,1)} = F_{(2,1)} + F_{(1,2)}$

Results: Combinatorics of crystal skeleton

2025 Brauner, Corteel, Daugherty, S.

Crystal skeleton $\mathrm{CS}(\lambda)$

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- **Vertices**
  1. Standard tableaux T
  2. Descent compositions α

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- **Vertices**
  1. Standard tableaux T
  2. Descent compositions α
- **Edges**
  1. Dyck pattern intervals: odd length intervals I with $|I| \geqslant 3$
  2. Cycles

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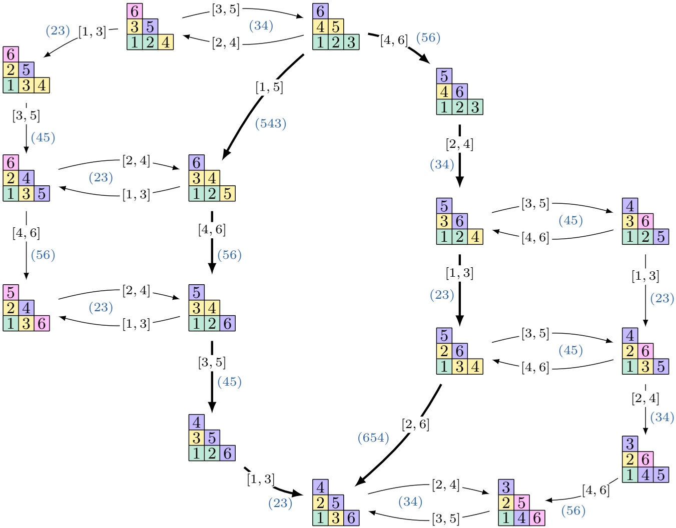

Edges: Dyck pattern intervals

Definition

- $\pi \in S_n$ permutation
- $I = [i, i+2m] \subseteq [n]$ interval of length $2m+1 \geqslant 3$

The interval I is a Dyck pattern interval of π if

$$P(\pi|_I) = \begin{array}{|c|c|c|c|c} \hline i+m+1 & i+m+2 & \cdots & i+2m & \\ \hline i & i+1 & \cdots & i+m-1 & \multicolumn{1}{c|}{i+m} \\ \hline \end{array}$$

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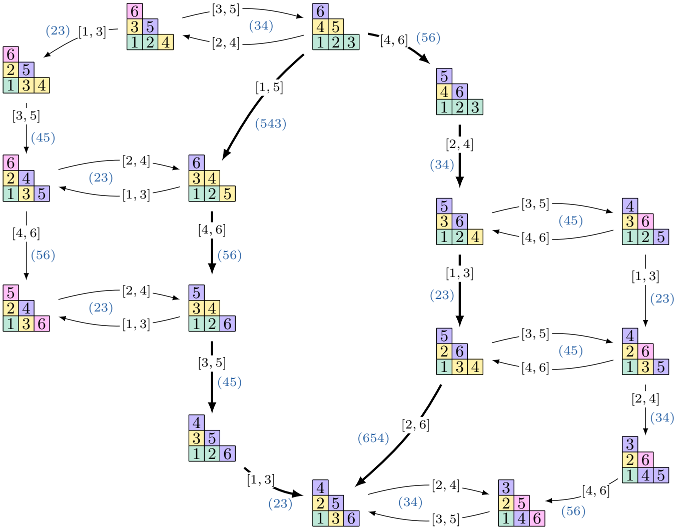
\end{array}$$

Example

$\pi = 524136$. The interval $I = [2,4]$ is a Dyck pattern interval since $\pi|_{[2,4]} = 243$ and

$$P(\pi|_{[2,4]}) = \begin{array}{|c|c}
\hline
4 & \multicolumn{1}{|c}{} \\
\hline
2 & \multicolumn{1}{c|}{3} \\
\hline
\end{array}$$

Edges: Dyck pattern intervals



Edges: Cycles

Definition

$$\text{cycle}(\pi|_I) := (m + \pi_p, m + \pi_p - 1, \ldots, \pi_p)$$

π_p is letter where f_i acts in destandardization and $I = [i, i + 2m]$

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$$I = [1,5] \qquad T = \begin{array}{|c|c|c|}\hline 6 & & \\ \hline 4 & 5 & \\ \hline 1 & 2 & 3 \\ \hline\end{array} \qquad \text{destand}(T) = \begin{array}{|c|c|c|}\hline 6 & & \\ \hline 2 & 2 & \\ \hline 1 & 1 & 1 \\ \hline\end{array} \qquad \pi = 645123$$

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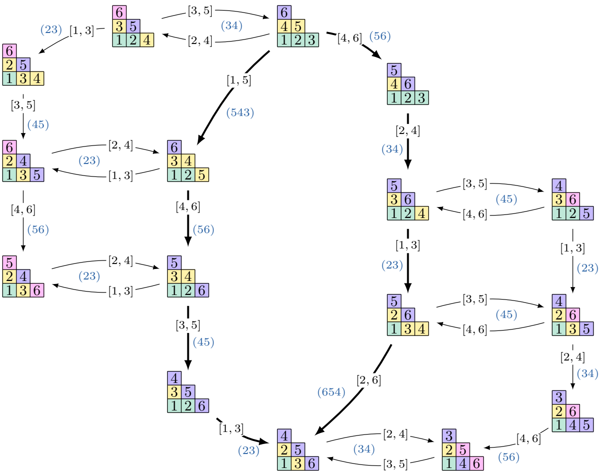
Dyck pattern interval $I = [1, 5]$

Cycle $\text{cycle}(\pi|_I) = (5, 4, 3)$

$$(5, 4, 3) \cdot T = T'$$

Edges: Cycles



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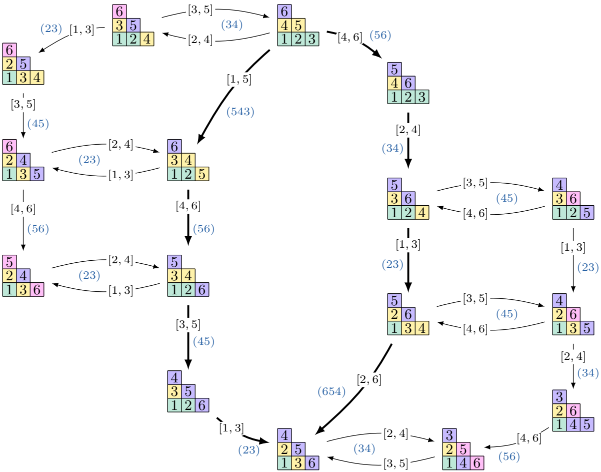

Reason 7

Crystal skeletons have a concrete combinatorial description.

Length of descent composition

Theorem (BCDS'25)

Edge in $\mathrm{CS}(\lambda)$

$$(T, \alpha) \xrightarrow{I} (T', \beta)$$

α *of length* $\ell \Rightarrow \beta$ *of length* $\ell - 1, \ell$ *or* $\ell + 1$

Properties: S_n-branching

Theorem (BCDS'25)

$$\mathrm{CS}(\lambda)_{[1,n-1]} \cong \bigoplus_{\lambda^-} \mathrm{CS}(\lambda^-)$$

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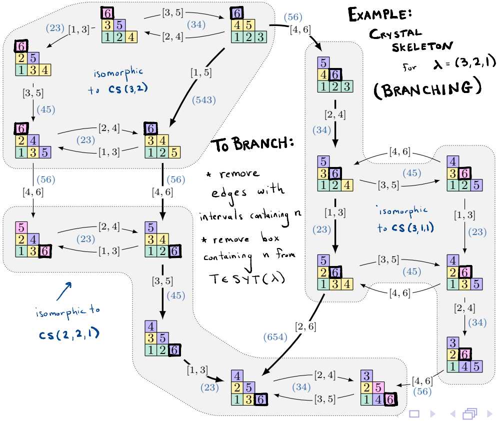

Properties: Subcrystal

Theorem (BCDS'25)

$\mathrm{CS}(\lambda)$ *contains crystal* $B(\lambda)$ *as a subgraph.*

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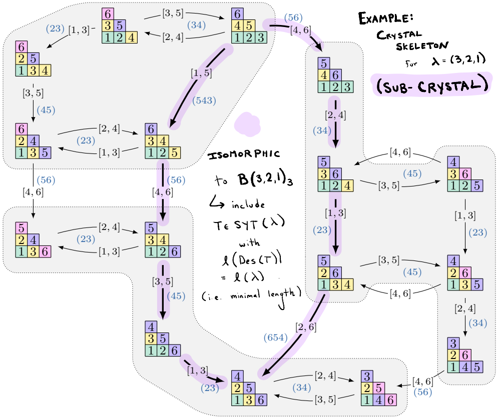

Properties: Dual equivalence subgraph

Theorem (BCDS'25 proving conjecture of Maas-Gariépy)

The dual equivalence graph $\mathrm{DE}(\lambda)$ *is a subgraph of* $\mathrm{CS}(\lambda)$.

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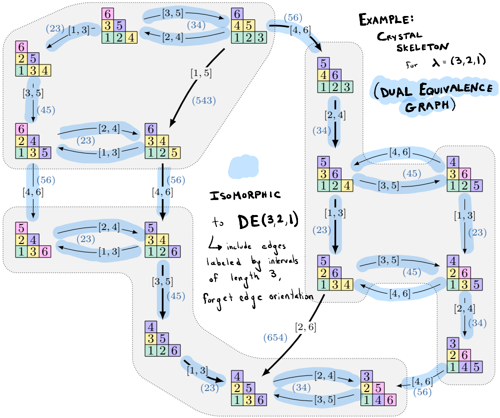

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Theorem (BCDS'25)

$\mathrm{CS}(\lambda)$ *is symmetric under Lusztig involution:* $\mathrm{CS}(\lambda) \cong \mathcal{L}_n(\mathrm{CS}(\lambda))$.

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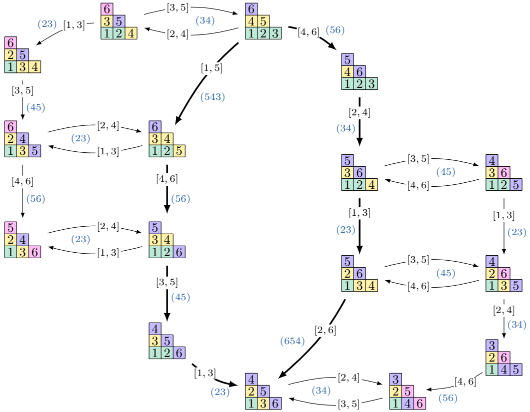

Axiomatic description

1. *GL_n*-**axioms**:
   - Strong Lusztig involution: $G \cong \mathcal{L}_n(G)$ and $G_{[1,n-1]} \cong \mathcal{L}_{n-1}(G_{[1,n-1]})$
   - Subcrystal
   - Fans

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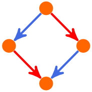
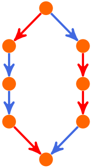
3. **Local axioms**:
   - Commutation relations (à la Stembridge): triangles, squares, pentagons, octagons + fans

Local characterization of crystal skeletons





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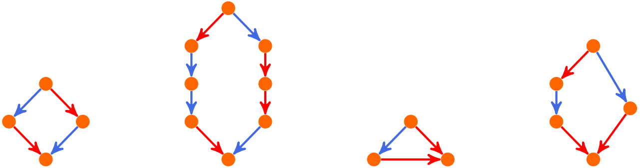
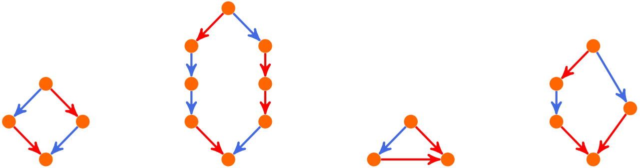


+ fans

Local characterization of crystal skeletons



+ fans

Reason 8

Crystal skeletons have many important properties and characterizations.

Outline

Motivation

Tiling and contraction

$$
\begin{array}{ccccc}
\text{Crystal } B(\lambda) & & \text{Crystal skeleton } \mathrm{CS}(\lambda) & & \\
\mathrm{char}B(\lambda) = s_\lambda & & \mathrm{char}\mathrm{CS}(\lambda) = s_\lambda & & \\
\cup\! \mid & \longrightarrow & \cup\! \mid & \longrightarrow & \text{Bruhat order} \\
\text{Quasicrystal } Q_\beta & & \text{Quasicrystal skeleton } \mathrm{QCS}_\alpha & & \\
\mathrm{char}Q_\beta = F_\beta & & \mathrm{char}\mathrm{QCS}_\alpha = \mathcal{YQS}_\alpha & &
\end{array}
$$

Based on joint work at ICERM with

Sarah Brauner, Zajj Daugherty, and Sarah Mason
preprint

Young quasisymmetric Schur function

2011 Haglund, Luoto, Mason, van Willigenburg
2013 Luoto, Mykytiuk, van Willigenburg

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$$\mathcal{YQS}_\alpha = \sum_{T \in \mathsf{SSYCT}(\alpha)} x^T$$

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$$\mathcal{YQS}_\alpha = \sum_{T \in \mathsf{SSYCT}(\alpha)} x^T$$

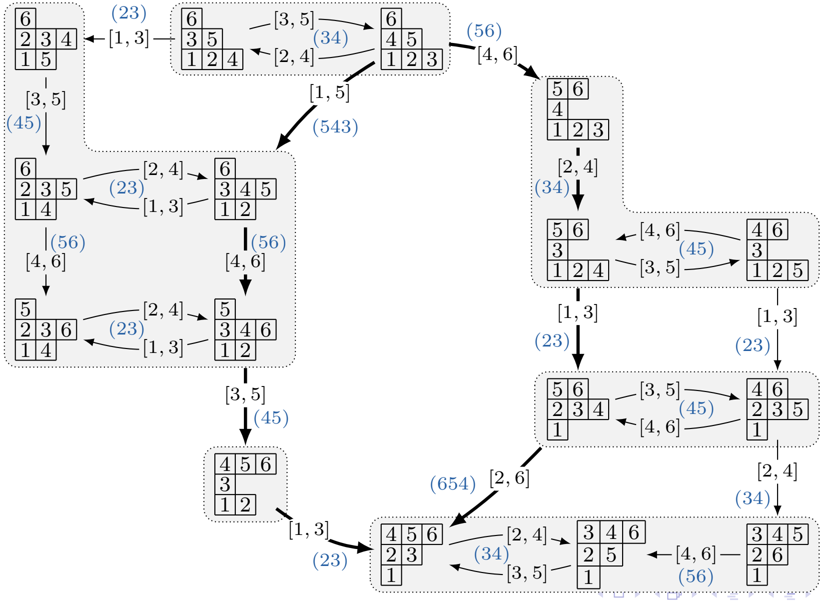
Definition

Semistandard Young composition tableau (SSYCT) of shape α:
Filling T of the cells of α such that

1. Rows are weakly increasing.
2. Leftmost column is strictly increasing.
3. Triple condition: If $a \geqslant b$ then $a > c$.

$$\begin{array}{|c|c|}\hline b & c \\ \hline \end{array}$$
$$\begin{array}{c|c|}\multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-2} & a \\ \cline{2-2}\end{array}$$

Quasicrystal skeleton



Applications

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Reason 9

Crystal skeletons potentially have important applications.

Thank you !

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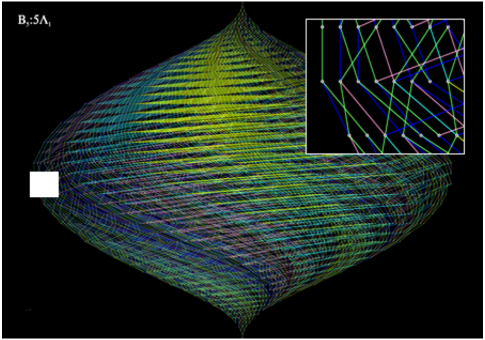

Reason 10

Crystals are beautiful!