IAP LECTURE JANUARY 28, 2000: THE ROGERS–RAMANUJAN IDENTITIES AT Y2K

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Abstract. The Rogers-Ramanujan identities have reached the ripe-old age of one hundred and five and are still the subject of active research. We will discuss their fascinating history, some of the number theory and combinatorics they encapture, and what they have to do with the 1998 Nobel Prize in physics.

1. The Rogers–Ramanujan identities

In this lecture you will be introduced to the Rogers–Ramanujan identities, you will learn some of their history and their fascinating relation to combinatorics and physics.

The Rogers–Ramanujan identities are

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})}, \tag{1.2}$$

where $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ for $n>0$ and $(q)_0 = 1$. (In the sequel we will mostly deal with (1.1); however all results have analogues for (1.2)). Although equality between sums and products expressed by these identities may appear rather obscure, you will hopefully be convinced by the end of the lecture how interesting they really are. We will not give proofs of (1.1) and (1.2) since they are quite involved, but instead discuss some results they imply.

What do the Rogers–Ramanujan identities mean? Basically, they give us two different expressions for the same function, a function in q. However, often the two sides are not viewed as functions, but rather as expressions for a *formal power series*.

What is a formal power series? Let us consider, for example, the function $f(x) = \frac{1}{1-x}$. Those of you who have taken 18.014 know that the Taylor or power series of this function for $-1 < x < 1$ is given by

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots . \tag{1.3}$$

This basically means that we can obtain the value of $f(x) = \frac{1}{1-x}$ when $-1 < x < 1$ up to a certain error by summing up sufficiently many terms on the right-hand side. To make the error smaller requires to take more terms into account. But to any

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given error it suffices to sum up a finite number of terms on the right-hand side. To see that $1 + x + x^2 + \cdots$ really amounts to $\frac{1}{1-x}$ observe that

$$\frac{1 - x^{m+1}}{1 - x} = 1 + x + x^2 + \cdots + x^m. \tag{1.4}$$

This equation can be verified by multiplying both sides by $1 - x$ and expanding the right-hand side. All terms except $1 - x^{m+1}$ cancel. Taking $m \to \infty$ in (1.4) yields (1.3) since $\lim_{m\to\infty} x^{m+1} = 0$ for $-1 < x < 1$.

Let us now use (1.3) to expand both sides of (1.1). The right-hand side becomes

$$\begin{aligned}\prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})} =& \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)} \cdots \\ =&(1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + q^7 \cdots) \\ &(1 + q^4 + q^8 + q^{12} + \cdots) \\ &(1 + q^6 + q^{12} + \cdots) \\ &\cdots \\ =&1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \cdots .\end{aligned} \tag{1.5}$$

Expanding the left-hand side of (1.1) in an analogous fashion we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} =& \frac{1}{(q)_0} + \frac{q}{(q)_1} + \frac{q^4}{(q)_2} + \frac{q^9}{(q)_3} + \cdots \\ =&1 + \frac{q}{(1-q)} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots \\ =&1 + q(1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + \cdots) \\ &+ q^4(1 + q + q^2 + q^3 + \cdots)(1 + q^2 + q^4 + \cdots) \\ &+ q^9(1 + q + q^2 + \cdots)(1 + q^2 + q^4 + \cdots)(1 + q^3 + q^6 + \cdots) \\ &+ \cdots \\ =&1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \cdots .\end{aligned}$$

As we see from the tedious calculations above the two sides of (1.1) yield the same power series. However, so far we have not worried about the convergence properties of these series. All manipulations we did were "formal" in the sense that we applied (1.3) without specifying for which ranges of q these manipulations are justified. That's why the above power series are labeled "formal power series". (Close scrutiny however reveals that the above power series converge for $-1 < q < 1$).

As we will see shortly, the coefficients in the power series corresponding to (1.1) are special numbers which have a number theoretic meaning. That is, writing either side as

$$(1.1) = \sum_{N=0}^{\infty} a_N q^N \tag{1.6}$$

the numbers $a_0, a_1, a_2, a_3, a_4, a_5, a_6 \ldots$ (which for (1.1) are $1, 1, 1, 1, 2, 2, 3, \ldots$) turn out to be the number of certain kinds of partitions. The sum $\sum_{N=0}^{\infty} a_N q^N$ is said to be the *generating function* of the sequence $a_0, a_1, a_2, \ldots$. But before going into detail let us disgress and indulge ourselves in the history of the Rogers–Ramanujan identities.

2. Some history

The Rogers–Ramanujan identities first appeared, together with a proof, in 1894 in a paper by Rogers [12]. However, his paper went practically unnoticed by the mathematics community at the time. In 1913 Ramanujan, an Indian genius who had no formal education in mathematics, wrote a letter to the English mathematician Hardy with several astonishing results. Among them were the identities (1.1) and (1.2), however without proof. Hardy was so amazed by Ramanujan's results that he invited him to England. However neither Ramanujan nor any of the mathematicians to whom Hardy had communicated Ramanujan's results were able to find a proof of the identities (1.1) and (1.2), so that they were published in 1916 in the book *Combinatory Analysis* by MacMahon [11] without proof.

In 1917 Ramanujan looked through some old Proceedings of the London Mathematical Society and came accidentally across Rogers' paper. Ramanujan must have been extremely surprised by his findings. He started a correspondence with Rogers which led to a joint paper [13] with a simplified proof of the identities (1.1) and (1.2). At the same time, cut off from England by the war, Schur rediscovered and proved the Rogers–Ramanujan identities [14]. His proof is quite different from the others and uses combinatorial methods. Because of his contribution some people refer to (1.1) and (1.2) as the Rogers–Schur–Ramanujan identities.

Since these early days, many new proofs and a vast number of generalizations of the Rogers–Ramanujan identities have been found. Because of the wealth of results it is not possible to give a complete account on all work related to the Rogers–Ramanujan identities here. Hence I will just mention some major breakthroughs. In 1951 Slater [15] compiled a list of 130 Rogers–Ramanujan-type identities using a method by Bailey [4]. The first infinite family of Rogers–Ramanujan-type identities was given by Andrews [1].

The debut of the Rogers–Ramanujan identities in physics was made in 1981 in a paper by Baxter [5] on the Hard Hexagon model. We will discuss some of Baxter's results in section 4. However, it was not until the 1990's that McCoy and his collaborators interpreted the Rogers–Ramanujan identities as the partition function of a physical system with quasiparticles obeying certain exclusion statistics [9, 10]. These exclusion statistics are related to the fractional statistics introduced by Haldane [8] which show up in the fractional quantum Hall effect for which the 1998 Nobel Prize in physics was awarded [16]! We will sketch the relation between the Rogers–Ramanujan identities and fractional statistics in section 5.

3. Partitions with restricted parts

We now wish to give an interpretation of the numbers a_N which occur as coefficients in the expansion (1.6). To this end let us consider a slight generalization of the expansion we made in (1.5).

Let S be a set of positive integers. We are interested in the expansion of the product $\prod_{n\in S}\frac{1}{1-q^n}$. Note that $\prod_{j=0}^{\infty}\frac{1}{(1-q^{5j+1})(1-q^{5j+4})}$ can be recovered by choosing S to be the set of all positive integers congruent to 1 or 4 modulo 5 since these are the numbers $5j+1$ and $5j+4$ for $j=0,1,2,\ldots$. Let us index the elements of

S as follows $S = \{s_1, s_2, s_3, \ldots\}$. Then, using (1.3), we obtain

$$\begin{aligned}\prod_{n\in S} \frac{1}{1-q^n} = \prod_{n\in S}(1+q^n+q^{2n}+q^{3n}+\cdots) \\ =&(1+q^{s_1}+q^{2s_1}+q^{3s_1}+\cdots) \\ &\times(1+q^{s_2}+q^{2s_2}+q^{3s_2}+\cdots) \\ &\times(1+q^{s_3}+q^{2s_3}+q^{3s_3}+\cdots) \\ &\cdots.\end{aligned}$$

This infinite product is to be expressed as a sum $\sum_{N=0}^{\infty} b_N q^N$. To obtain the term q^N from the product we need to pick one summand in each of the factors such that the exponents add up to N. From the first factor we get an exponent of the form $n_1 s_1$ with $n_1 \in \{0, 1, 2, \ldots\}$. From the second factor we get an exponent of the form $n_2 s_2$ with $n_2 \in \{0, 1, 2, \ldots\}$, and so on. Hence b_N counts the number of ways N can be written as $n_1 s_1 + n_2 s_2 + n_3 s_3 + \cdots$ with $n_1, n_2, n_3, \ldots \geq 0$.

If $s_i = i$, so that $S = \{1, 2, 3, \ldots\}$ is the set of all positive integers, b_N is the number of ways one can write N as a sum of positive integers. For example, if $N = 4$ we get

$$\begin{aligned}4 =&1+1+1+1 \\ =&2+1+1 \\ =&2+2 \\ =&3+1 \\ =&4.\end{aligned} \tag{3.1}$$

A collection of positive integers which sums up to N is called a *partition* of N. Hence the partitions of 4 are given by $(1, 1, 1, 1)$, $(2, 1, 1)$, $(2, 2)$, $(3, 1)$ and (4). Since there are five of them we have $b_4 = 5$. Each element in a partition is called a part. For example the partition $(2, 1, 1)$ has three parts given by 2, 1 and 1.

When $S = \{s_1, s_2, s_3, \ldots\}$ the coefficient b_N is still the number of partitions of N; however there is the additional restriction that the parts of the partitions have to be an element of S.

Recall that for the expansion (1.6) the set S is the set of all positive integers congruent to 1 or 4 modulo 5. For example, the partitions of 6 with parts congruent to 1 or 4 modulo 5 are the following:

$$(1, 1, 1, 1, 1, 1), \quad (4, 1, 1) \quad \text{and} \quad (6).$$

There are three of them which is indeed a_6.

The above arguments imply the following theorem.

Theorem 1. *The right-hand side of* (1.1) *is the generating function of partitions with parts congruent* 1 *or* 4 *modulo* 5*.*

This theorem indeed shows that the coefficients $a_0, a_1, a_2, \ldots$ of the expansion (1.6) have a combinatorial meaning. A good introductory book on the theory of partitions is [2].

4. The Hard Hexagon model

It turns out that partition theorists and combinatorialists are not the only ones interested in counting partitions. Physicists are also interested in precisely the

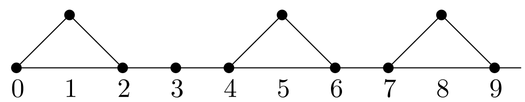

FIGURE 1. A path of length 9

same counting problems, since applied to models of microscopic particles, they allow the calculation of macroscopic properties such as specific heat, resistance, conductivity etc. In fact, as mentioned earlier the physicist Baxter encountered the Rogers–Ramanujan identities in his study of the Hard Hexagon model in 1981 [5, 6]. The Hard Hexagon model is a two-dimensional model in statistical mechanics which exhibits a phase transition. In order to study the properties of this phase transition Baxter reduced the two-dimensional problem to a one-dimensional problem which is of interest to us here.

Consider $L+1$ points on a line labeled by $i = 0, 1, 2, \ldots, L$. Assign to each point a height variable σ_i which takes on the values 0 or 1. In addition the height variables satisfy the restrictions $\sigma_0 = \sigma_L = 0$ and $\sigma_i \sigma_{i+1} = 0$. An allowed configuration of height variables for a given length L is called a *path* of length L. One can illustrate a path graphically by drawing all points (i, σ_i) and connecting adjacent points by straight lines. An example for a path with $L = 9$ is given in figure 1. The condition $\sigma_i \sigma_{i+1} = 0$ requires that the paths consist of a certain number of non-overlapping triangles. To each path p one may assign an energy $E(p)$ by summing up the positions of the peaks, that is

$$E(p) = \sum_{j=1}^{L} j \sigma_j.$$

The energy of the path in figure 1 is $E(p) = 1 + 5 + 8 = 14$. Let us now consider the function

$$F(L) = \sum_p q^{E(p)} \tag{4.1}$$

where the sum is over all paths of length L.

Our aim is to find an expression for $F(L)$ and to relate it to the left-hand side of (1.1). To this end let us work out some properties of $F(L)$. Recall that $\sigma_L = 0$. Hence the last step of the path from point $(L-1, \sigma_{L-1})$ to point $(L, 0)$ is either a horizontal line (if $\sigma_{L-1} = 0$) or it goes down (if $\sigma_{L-1} = 1$). In the former case the energy does not change if we remove the last step. Hence the contribution to $F(L)$ from all paths with the last step horizontal is $F(L-1)$. If $\sigma_{L-1} = 1$ the condition $\sigma_{L-2}\sigma_{L-1} = 0$ requires that $\sigma_{L-2} = 0$. Removing the last two steps of the path changes the energy by $L-1$ since there is a peak at $L-1$. Hence the contribution from these paths to $F(L)$ is $q^{L-1}F(L-2)$. Altogether this yields the following recursion relation

$$F(L) = F(L-1) + q^{L-1} F(L-2). \tag{4.2}$$

$F(L)$ is completely determined by this recurrence and the initial condition $F(0) = F(1) = 1$. Note that at $q = 1$ the recurrence (4.2) is precisely the recursion relation for the Fibonacci numbers.

A solution to (4.2) can be given in terms of q-binomial coefficients. These are q-deformations of the usual binomial coefficients $\binom{M}{m} = \frac{M!}{m!(M-m)!}$ and are defined as

$$\begin{bmatrix} M \\ m \end{bmatrix} = \frac{(q)_M}{(q)_m (q)_{M-m}} \tag{4.3}$$

for $0 \le m \le M$ and zero otherwise. The q-binomial $\left[\begin{smallmatrix} M \\ m \end{smallmatrix}\right]$ reduces to the binomial coefficient $\binom{M}{m}$ as $q \to 1$ (this can be shown using l'Hôpital's rule). For $q = 1$ the following result is probably familiar to you.

Lemma 2. *For $M > 0$ the q-binomial coefficients satisfy the recurrence*

$$\begin{bmatrix} M \\ m \end{bmatrix} = \begin{bmatrix} M-1 \\ m \end{bmatrix} + q^{M-m} \begin{bmatrix} M-1 \\ m-1 \end{bmatrix}. \tag{4.4}$$

Proof. We begin with the right-hand side of (4.4), insert the definition (4.3) and put everything over a common denominator

$$\begin{aligned} \begin{bmatrix} M-1 \\ m \end{bmatrix} + q^{M-m} \begin{bmatrix} M-1 \\ m-1 \end{bmatrix} &= \frac{(q)_{M-1}}{(q)_m (q)_{M-1-m}} + q^{M-m} \frac{(q)_{M-1}}{(q)_{m-1} (q)_{M-m}} \\ &= \frac{(q)_{M-1}}{(q)_m (q)_{M-m}} (1 - q^{M-m} + q^{M-m}(1 - q^m)) \\ &= \frac{(q)_{M-1}}{(q)_m (q)_{M-m}} (1 - q^M) = \begin{bmatrix} M \\ m \end{bmatrix}. \end{aligned}$$

□

In fact the recurrence (4.4) reduces to

$$\binom{M}{m} = \binom{M-1}{m} + \binom{M-1}{m-1}$$

for $q = 1$ which is the defining relation of the Pascal triangle.

Now define

$$P(L) := \sum_{n=0}^{\infty} q^{n^2} \begin{bmatrix} L-n \\ n \end{bmatrix}. \tag{4.5}$$

Theorem 3. *For $L \ge 0$, $F(L) = P(L)$.*

Proof. To prove $P(L) = F(L)$ it suffices to show that $P(0) = P(1) = 1$ and that $P(L) = P(L-1) + q^{L-1} P(L-2)$. For $L = 0$ or 1 the only nonzero contribution from the sum over n in (4.5) comes from the term $n = 0$, so that indeed $P(0) = P(1) = 1$.

To prove the recurrence we apply (4.4) to (4.5) and obtain

$$\sum_{n=0}^{\infty} q^{n^2} \left[\begin{smallmatrix} L-n \\ n \end{smallmatrix}\right] = \sum_{n=0}^{\infty} q^{n^2} \left[\begin{smallmatrix} L-n-1 \\ n \end{smallmatrix}\right] + \sum_{n=1}^{\infty} q^{n^2+L-2n} \left[\begin{smallmatrix} L-n-1 \\ n-1 \end{smallmatrix}\right].$$

(Note that the application of (4.4) for $n \ge L$ is ok since in this case both sides yield 0). The first sum is $P(L-1)$. To see that the second sum equals $q^{L-1} P(L-2)$ we make the substitution $n = m + 1$

$$\sum_{n=1}^{\infty} q^{n^2+L-2n} \left[\begin{smallmatrix} L-n-1 \\ n-1 \end{smallmatrix}\right] = \sum_{m=0}^{\infty} q^{m^2+L-1} \left[\begin{smallmatrix} L-m-2 \\ m \end{smallmatrix}\right] = q^{L-1} P(L-2).$$

□

Since $F(L)$ at $q=1$ satisfies the Fibonacci recurrence and $F(L)=P(L)$, we can interpret the explicit expression (4.5) as a q-deformation of the Fibonacci numbers.

The expression (4.5) looks similar to the left-hand side of (1.1), but is not quite the same. Note that, canceling the common factor $(q)_{L-2n}$, we obtain

$$\begin{bmatrix} L-n \\ n \end{bmatrix} = \frac{(q)_{L-n}}{(q)_n (q)_{L-2n}} = \frac{(1-q^{L-2n+1})(1-q^{L-2n+2})\cdots(1-q^{L-n})}{(q)_n}.$$

Let us fix n. Then for $-1<q<1$ all terms $q^{L-2n+k} \to 0$ as $L \to \infty$ so that

$$\lim_{L\to\infty} \begin{bmatrix} L-n \\ n \end{bmatrix} = \frac{1}{(q)_n}. \tag{4.6}$$

Hence the left-hand side of (1.1) equals

$$\lim_{L\to\infty} P(L) = \lim_{L\to\infty} F(L).$$

What do we learn from this result? Well, we know by (4.1) that $F(L)$ is the sum over all paths p of length L weighted by the energy function $E(p)$. The coefficient of q^N is hence the number of all paths with energy N. Recall that the energy is the sum of the positions of the peaks. Because of the restriction $\sigma_i \sigma_{i+1} = 0$ the peaks have to be at least two apart. Hence the number of paths with energy N is equal to the number of partitions for which the difference between any two parts is at least two. This proves the following theorem.

Theorem 4. *The left-hand side of* (1.1) *is the generating function of partitions for which the difference between any two parts is at least two.*

Let us check this theorem for $N=6$. The partitions of 6 with the difference between any two parts at least two are

$$(4,2), \quad (5,1) \quad \text{and} \quad (6).$$

Hence as asserted $a_6 = 3$.

Together with Theorem 1 this immediately yields the following corollary.

Corollary 5. *The number of partitions of an integer N into parts in which the difference between any two parts is at least 2 is the same as the number of partitions of N into parts congruent to* 1 *or* 4 *modulo* 5.

This corollary is a remarkable and highly non-trivial result. At first sight the partitions with restrictions on the difference of any two parts and the partitions with the modular restrictions have nothing to do with each other. Recall again that for $N=6$ the first set consist of the partitions $(1,1,1,1,1,1)$, $(4,1,1)$ and (6) whereas the second set contains the partitions $(4,2)$, $(5,1)$ and (6). The theorem asserts that the cardinality of both sets is equal.

5. Fractional statistics

In physics *statistics* refers to certain properties of particles. There are two well-known types of particles, *bosons* and *fermions*. Bosons are characterized by the property that each physical state can be occupied by an arbitrary number of particles. For fermions, on the other hand, each state can be occupied by at most one particle.

Let us consider a physical model which has single particle states of energies $1, 2, 3, 4, \ldots$. The partition function is the sum over all possible particle configurations c of the system weighted by the total energy $H(c)$, that is $\sum_c q^{H(c)}$. The total energy is just the sum of the all single particle energies.

If the particles are bosons then each single particle state can be occupied by an arbitrary number of single particles. Hence, if we want to determine how many possible particle configurations there are for a given energy N, we need to find the number of ways N can be written as a sum of positive integers (these are the single particle energies). But this is nothing but the number of partitions of N. As we have seen in section 3 the generating function of all partitions is $\prod_{n=1}^{\infty} \frac{1}{1-q^n}$. In fact there also exists an identity in this case given by

$$\sum_{m=0}^{\infty} \frac{q^m}{(q)_m} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{5.1}$$

If the particles are fermions each single particle state can be occupied by at most one particle. In this case the coefficient of q^N in the partition function is the number of partitions of N into distinct parts. Similar to section 3 it can be shown that the generating function of partitions with distinct parts is $\prod_{n=1}^{\infty}(1+q^n)$. The corresponding identity is

$$\sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} = \prod_{n=0}^{\infty}(1+q^n). \tag{5.2}$$

These identities for bosons and fermions look similar to the Rogers–Ramanujan identities. This immediately raises the question what kind of particles the Rogers–Ramanujan identities represent? To answer this let us look back at section 4 where we considered paths. The graphical illustration (see for example figure 1) suggests to interpret each triangle in the path as a particle. Since the particles cannot overlap each particle occupies two states. This means that adding a particle to the system reduces the number of available states by two. This differs from the boson or fermion statistics. For bosons the addition of a particle does not reduce the number of available states (since each state can be occupied by an arbitrary number of particles). For fermions the addition of a particle reduces the number of available states by one (since each state can be occupied by at most one particle).

Nowadays there exist a vast number of generalizations of the identities (1.1), (1.2), (5.1) and (5.2). It has been shown [9, 10] that the analogue of the left-hand side of these identities carries the information of the statistics of the underlying particles. In many of the known examples a particle occupies only a fraction of a single particle state. This may seem odd, however recently quasiparticles possessing exactly these properties have been observed in fractional quantum Hall samples. The discovery of the fractional quantum Hall effect was honored in 1998 by the Nobel prize in physics [16].

To conclude the lecture let me cite Dyson's famous paper "Missed opportunities" [7] (with which Andrews' lecture [3] series actually begins):

> "As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce... I shall examine in detail some examples of missed opportunities,

occasions on which mathematicians and physicists lost chances of making discoveries by neglecting to talk to each other."

The Rogers–Ramanujan identities are an example where the "divorce" between mathematics and physics seems to be overcome at last!

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[16] see for example `http://mirror.nobel.ki.se/laureates/physics-1998.html`.

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