Lecture 1: Crystals for stable Grothendieck polynomials

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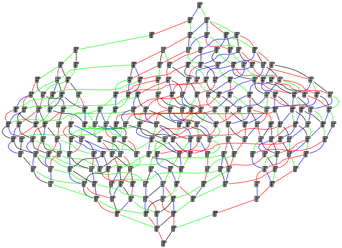


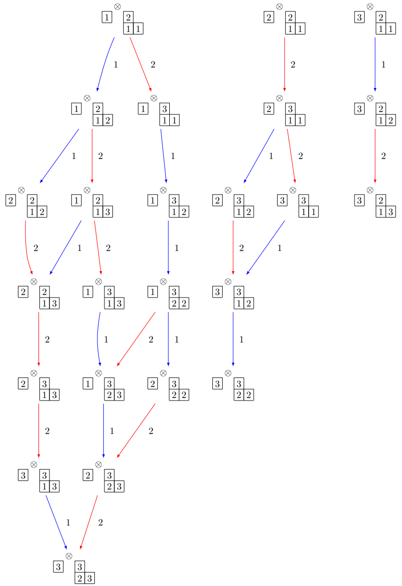
Osaka, March 5, 2023

This is based joint work with
Jennifer Morse (2016) & Jennifer Morse, Jianping Pan, Wencin Poh (2020)

Outline

Crystal graphs



- The generating function

$$\sum_{\text{vertex } b} \mathbf{x}^{\text{weight}(b)}$$

  is the character of the crystal.

- The character of each connected component is a Schur function

$$s_\lambda(\mathbf{x}) = \sum_{T \in \text{SSYT}(\lambda)} \mathbf{x}^{\text{weight}(T)}$$

  where λ is the weight of the highest element.

Crystal operators

Action of crystal operators e_i, f_i on words/tableaux:

1. Consider letters i and $i+1$ in row reading word of the tableau
2. Successively "bracket" pairs of the form $(i+1, i)$
3. Left with word of the form $i^r(i+1)^s$

$$e_i(i^r(i+1)^s) = \begin{cases} i^{r+1}(i+1)^{s-1} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

$$f_i(i^r(i+1)^s) = \begin{cases} i^{r-1}(i+1)^{s+1} & \text{if } r > 0 \\ 0 & \text{else} \end{cases}$$

Outline

Stable Schubert polynomials F_w

- restriction: $\mathfrak{S}_{1_m \times w} \longrightarrow$ Stanley symmetric functions F_w for $w \in S_n$
- for 321-avoiding w,

$$F_w = s_{\nu/\mu} = \sum_\lambda c^\nu_{\lambda\mu}\, s_\lambda$$

- symmetric and Schur positive (Stanley 1984, Edelman, Greene 1987)

$$F_w = \sum_\lambda a_{w\lambda}\, s_\lambda$$

- coefficient of $x_1 x_2 \cdots x_r$ counts reduced words of w

$$S_n = \langle s_1, \ldots, s_{n-1} \rangle \qquad s_i s_j = s_j s_i \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad s_i^2 = id$$

$$(3,2,1,4) = s_1 s_2 s_1 = s_2 s_1 s_2 = s_3 s_3 s_1 s_2 s_1$$

Stable Schubert polynomials

$$F_w = \sum_{v^r \cdots v^1 = w} x_1^{\ell(v^1)} \cdots x_r^{\ell(v^r)}$$

Decreasing factorization of w

1. w is the product of permutations $v^r \cdots v^1$
2. each v^i has a strictly decreasing reduced word
3. $\ell(w) = \ell(v^r) + \cdots + \ell(v^1)$

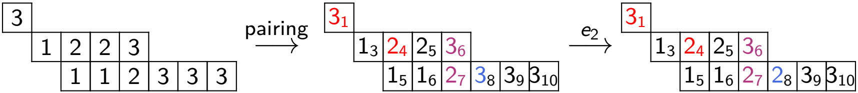
$w = (2,1,4,3) = s_1 s_3 = s_3 s_1$:

$$(s_1)(s_3) \longrightarrow x_1 x_2$$
$$(s_3)(s_1) \longrightarrow x_1 x_2$$
$$()(s_3 s_1) \longrightarrow x_1^2$$
$$(s_3 s_1)() \longrightarrow x_2^2$$

$$F_{(2,1,4,3)} = 2\, x_1 x_2 + x_1^2 + x_2^2$$

Crystal operators on factorizations – residue map

Label cells diagonally



$$\underbrace{(10\ 9\ 8\ 6\ 1)}_{\text{label of 3's}}\ \underbrace{(7\ 5\ 4)}_{\text{label of 2's}} \xrightarrow{\text{pairing}} \underbrace{(10\ 9\ 8\ 6\ 1)}_{\text{label of 3's}}\ \underbrace{(7\ 5\ 4)}_{\text{label of 2's}}$$

$$\xrightarrow{e_2} \underbrace{(10\ 9\ 6\ 1)}_{\text{label of 3's}}\ \underbrace{(8\ 7\ 5\ 4)}_{\text{label of 2's}}$$

operator e_i

from big to small:
pair $x \in$ 3's with smallest $y \in$ 2's that is bigger than x
delete smallest unpaired $z \in$ 3's and add $z - t$ to 2's

$$(9\ 8\ 6\ 5\ 4\ 1)(9\ 6\ 5\ 2\ 1) \to (9\ 8\ 5\ 4\ 1)(9\ 6\ 5\ 4\ 2\ 1)$$

Crystal Theorem

Definition

Fix $w \in S_n$.

Graph $B(w)$

1. vertices are decreasing factorizations of w
2. edges are imposed and colored by f_i, e_i
3. highest weights are vertices with no unpaired entries

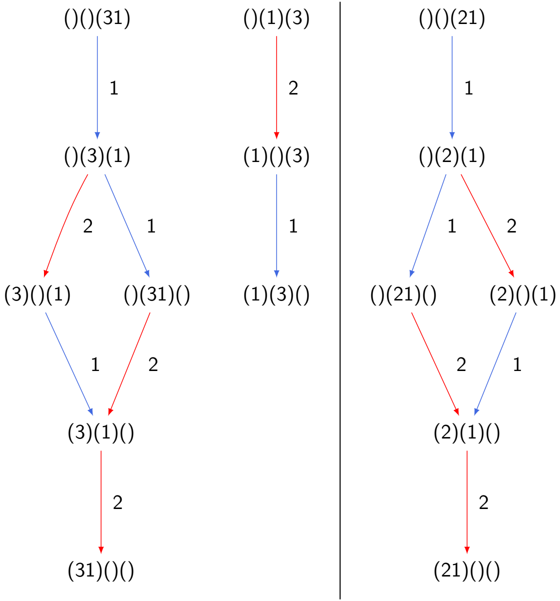
Theorem (with Morse; 2016)

$B(w)$ is a crystal graph of type A_ℓ

Proof

Checking Stembridge local axioms

Examples



Schur expansion

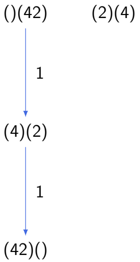
Fix $w \in S_n$

Theorem (with Morse; 2016)

$$F_w = \sum_{\lambda} a_{w\lambda}\, s_\lambda$$

$a_{w\lambda}$ counts highest weights $v^r \cdots v^1$ of $B(w)$ with $(\ell(v^1), \ldots, \ell(v^r)) = \lambda$

In S_5:



$\Longrightarrow F_{s_2 s_4} = s_2 + s_{11}$

Edelman-Greene insertion

Theorem (with Morse; 2016)

For any permutation $w \in S_n$, the crystal isomorphism

$$B(w) \cong \bigoplus_{\lambda} B(\lambda)^{\oplus a_{w\lambda}}$$

is explicitly given by the Edelman-Greene insertion $\varphi^{Q}_{\mathrm{EG}}(v^{\ell} \cdots v^{1}) = Q$:

$$\varphi^{Q}_{\mathrm{EG}} \circ e_i = e_i \circ \varphi^{Q}_{\mathrm{EG}}$$


Emil
i Lönneberga

Outline

Motivation: Schubert Calculus

Polynomial Representatives for Schubert Cells

	Grassmannian $\mathbb{G}_{m,n}$	**Flag Varieties** Fl_n
Cohomology	s_λ	$\mathfrak{S}_w \to F_w$
K-**theory**	$\mathfrak{G}_\lambda$	$\mathfrak{G}_w$

Grassmannian Grothendieck polynomials: $\mathfrak{G}_\lambda$ Lascoux, Schützenberger 1982
Stable Grothendieck polynomials: $\mathfrak{G}_w$ Fomin, Kirillov 1994

Combinatorial Approach?

Combining:

- Crystal structure on decreasing factorizations for F_w (Morse, S. 2016)
- Crystal structure for $\mathfrak{G}_\lambda$ on set-valued tableaux (Monical & Pechenik & Scrimshaw 2018)

0-Hecke Monoid

Definition

0-Hecke monoid $\mathcal{H}_0(n)$:
monoid of all finite words in $[n] := \{1, 2, \ldots, n\}$ such that

$$pp \equiv p, \quad pqp \equiv qpq \quad \text{for all } p, q \in [n]$$
$$pq \equiv qp \quad \text{if } |p - q| > 1$$

Examples

- $2112 \equiv 212 \equiv 121$
- $2121 \equiv 1211 \equiv 121 \equiv 212$
- $31312 \equiv 3132 \equiv 312 \equiv 132$

Decreasing factorizations in $\mathcal{H}_0(n)$

Definition

A decreasing factorization of $w \in \mathcal{H}_0(n)$ into m factors is a product of decreasing factors

$$\mathbf{h} = h^m \dots h^2 h^1$$

such that $\mathbf{h} \equiv w$ in $\mathcal{H}_0(n)$.

$\mathcal{H}_w^m$ = set of decreasing factorizations of w in $\mathcal{H}_0(n)$ with m factors

Example

Decreasing factorizations for $132 \in \mathcal{H}_0(3)$ of length 5 with 3 factors:

$$(31)(31)(2) \qquad (31)(32)(2) \qquad (31)(1)(32)$$
$$(31)(3)(32) \qquad (1)(31)(32) \qquad (3)(31)(32)$$

Stable Grothendieck polynomials for w

Definition

Stable Grothendieck polynomial (or K-Stanley symmetric function):

$$\mathfrak{G}_w(\mathbf{x},\beta) = \sum_{h^m \ldots h^2 h^1 \in \mathcal{H}_w^m} \beta^{\ell(h^1)+\cdots+\ell(h^m)-\ell(w)} x_1^{\ell(h^1)} \ldots x_m^{\ell(h^m)}$$

where $\ell(w)$ is the length of any reduced word of w.

Example

$w = 132 \in \mathcal{H}_0(3)$

Reduced Hecke words $132, 312$

Decreasing factorizations for constant term:

$(31)(2), (1)(32)$ $(3)(1)(2), (1)(3)(2)$

$$\beta^0 : \; (x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2) + 2x_1x_2x_3 = s_{21}$$

Schur positivity

Schur positivity (Fomin, Greene 1998)

$$\mathfrak{G}_w(\mathbf{x}, \beta) = \sum_{\lambda} \beta^{|\lambda| - \ell(w)} g_w^{\lambda} s_{\lambda}(\mathbf{x})$$

$$g_w^{\lambda} = |\{ T \in \mathrm{SSYT}^n(\lambda') | \text{ column reading of } T \equiv w \}|$$

Example

$$\mathfrak{G}_{132}(\mathbf{x}, \beta) = s_{21} + \beta(2s_{211} + s_{22}) + \beta^2(3s_{2111} + 2s_{221}) + \cdots$$

321-avoiding Hecke words (braid-free)

Definition

$w \in \mathcal{H}_0(n)$ is 321-avoiding if none of the reduced expressions for w contain a consecutive subword of the form $i\ i+1\ i$ for any $i \in [n-1]$.

Examples

- $1321 \equiv 3121 \equiv 3212$ is not 321-avoiding
- $22132 \equiv 2132 \equiv 2312$ is 321-avoiding

Definition

$\mathcal{H}^{m,\star}$ = set of decreasing factorizations into m factors for 321-avoiding w

Example

- $(\ )(1)(21) \in \mathcal{H}^3, \notin \mathcal{H}^{3,\star}$
- $(31)(2) \in \mathcal{H}^{2,\star}$
- $(2)(21)(32) \in \mathcal{H}^{3,\star}$

$\star$-Crystal on $\mathcal{H}^{m,\star}$ (Morse, Pan, Poh, S.)

Bracketing rule on $h^m \ldots h^{i+1}\, h^i \ldots h^1$

1. Start with the **largest** letter b in h^{i+1}, pair it with the smallest $a \geqslant b$ in h^i. If there is no such a, then b is unpaired.
2. Proceed in decreasing order in h^{i+1}, ignore previously paired letters.

Crystal operator $f_i^\star$, x : largest unpaired letter in h^i

- If $x+1 \in h^i \cap h^{i+1}$, then remove $x+1$ from h^i, add x to h^{i+1}.
- Otherwise, remove x from h^i and add x to h^{i+1}.

Example

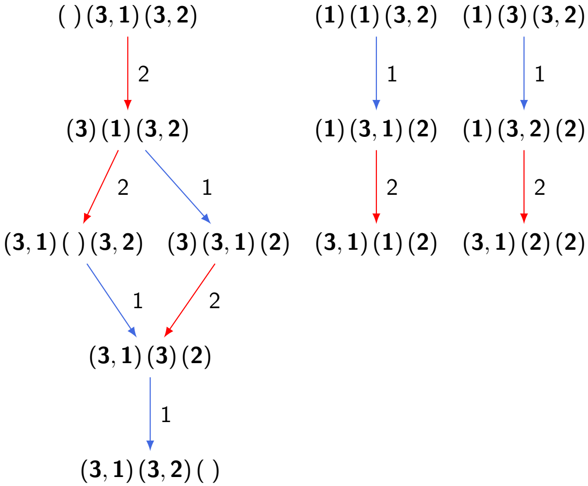
- $(1)(32) \xrightarrow{\text{bracket}} (1)(32) \xrightarrow{f_1^\star} (31)(2)$
- $(7532)(621) \xrightarrow{\text{bracket}} (7532)(621) \xrightarrow{f_1^\star} (75321)(61)$

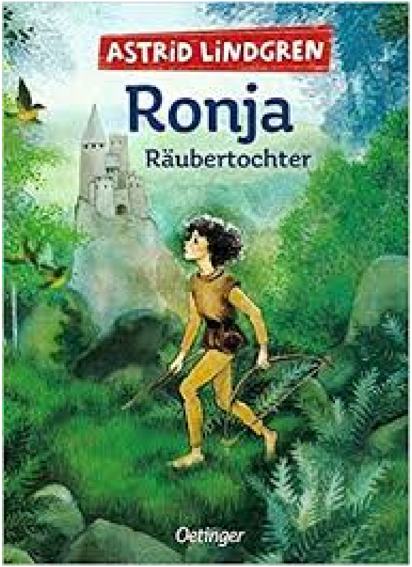
Vertices and edges

$w = 132, \ \beta^1$

1. $(3,1)(3,2)(\,)$
2. $(3,1)(1)(2)$
3. $(3,1)(2)(2)$
4. $(3,1)(3)(2)$
5. $(1)(3,1)(2)$
6. $(1)(3,2)(2)$
7. $(3)(3,1)(2)$
8. $(3,1)(\,)(3,2)$
9. $(1)(1)(3,2)$
10. $(1)(3)(3,2)$
11. $(3)(1)(3,2)$
12. $(\,)(3,1)(3,2)$

$$\mathfrak{G}_{132}(\mathbf{x}, \beta) = s_{21} + \beta(2s_{211} + s_{22}) + \beta^2(3s_{2111} + 2s_{221}) + \cdots$$

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Outline

Grothendieck polynomials for skew shapes

$$\mathfrak{G}_{\nu/\lambda}(\mathbf{x};\beta) = \sum_{T \in \mathrm{SVT}(\nu/\lambda)} \beta^{\mathrm{ex}(T)} \mathbf{x}^{\mathrm{wt}(T)} \qquad \text{(Buch 2002)}$$

$\mathrm{SVT}(\nu/\lambda)$ = set of semistandard set-valued tableaux of shape ν/λ

Excess in T is $\mathrm{ex}(T)$

Semistandard set-valued tableaux $\mathrm{SVT}(\nu/\lambda)$

Fill boxes of skew shape ν/λ with nonempty sets. Semistandardness:

C	
A	B

$\max(A) \leqslant \min(B)$, $\max(A) < \min(C)$

Example (Which one is a valid filling?)

✓

34	45	
	12	25

34	35	
	12	456

2	35	
	14	56

Crystal structure on SVT (Monical, Pechenik, Scrimshaw)

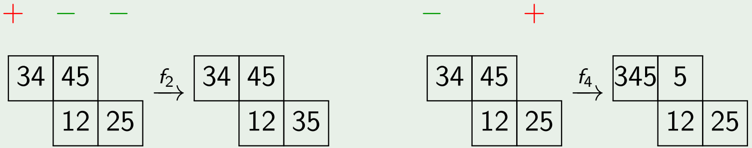
Signature rule

Assign $-$ to every column of T containing an i but not an $i+1$.
Assign $+$ to every column of T containing an $i+1$ but not an i.
Successively pair each $+$ that is adjacent to a $-$.

Crystal operator f_i

- changes the rightmost unpaired i $-$ to $i+1$, except
- if its right neighbor contains both $i, i+1$, then *move* the i over and turn it into $i+1$

Example



Residue map as a crystal isomorphism

Theorem (Morse, Pan, Poh, S. 2020)

The crystal on skew semistandard set-valued tableaux and the crystal on decreasing factorizations $\mathcal{H}^{m,\star}$ intertwine under the residue map. That is, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{SVT}^m(\lambda/\mu) & \xrightarrow{\text{res}} & \mathcal{H}^{m,\star} \\ \downarrow f_k & & \downarrow f_k^\star \\ \mathrm{SVT}^m(\lambda/\mu) & \xrightarrow{\text{res}} & \mathcal{H}^{m,\star}. \end{array}$$

Example

$$\begin{array}{ccc} \begin{array}{|c|c} \hline 3_1 & \\ \hline 1_2 & \multicolumn{1}{|c|}{123_3} \\ \hline \end{array} & \xrightarrow{\text{res}} & (31)(3)(32) \\ \downarrow f_1 & & \downarrow f_1^\star \\ \begin{array}{|c|c} \hline 3_1 & \\ \hline 12_2 & \multicolumn{1}{|c|}{23_3} \\ \hline \end{array} & \xrightarrow{\text{res}} & (31)(32)(2) \end{array}$$

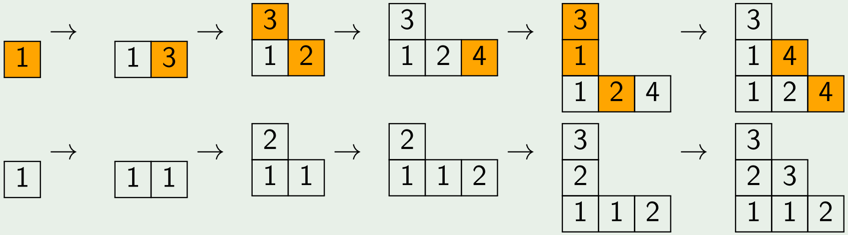
⋆-Insertion

Insert x into row R of a transpose of a semistandard tableau

1. Try to append x to the right of R (terminate and record)
2. $x \notin R$, bump the minimal $z > x$ (proceed to the next row)
3. $x \in R$, proceed to next row with y minimal such that $[y, x] \subseteq R$

Example

$$\mathbf{h} = (42)(42)(31) = \begin{bmatrix} \mathbf{3} & \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{4} & \mathbf{2} & \mathbf{4} & \mathbf{2} & \mathbf{3} & \mathbf{1} \end{bmatrix}$$



Association with $\star$-crystal

Theorem (Morse, Pan, Poh, S. 2020)

The following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \mathrm{SSYT}^m \\ \big\downarrow f_i^\star & & \big\downarrow f_i \\ \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \mathrm{SSYT}^m \end{array}$$

Example

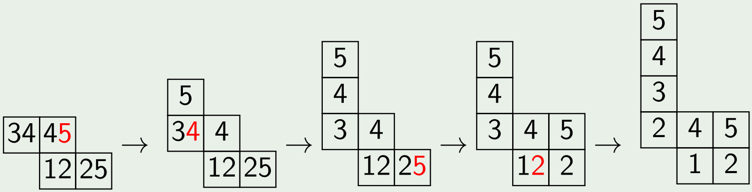
$$\begin{array}{ccc} (42)(42)(31) & \xrightarrow{\quad\star\quad} & \left(\begin{array}{lll} 3 & & \\ 1 & 4 & \\ 1 & 2 & 4 \end{array} , \begin{array}{lll} 3 & & \\ 2 & 3 & \\ 1 & 1 & 2 \end{array} \right) \\ \big\downarrow f_1^\star & & \big\downarrow f_1 \\ (42)(421)(3) & \xrightarrow{\quad\star\quad} & \left(\begin{array}{lll} 3 & & \\ 1 & 4 & \\ 1 & 2 & 4 \end{array} , \begin{array}{lll} 3 & & \\ 2 & 3 & \\ 1 & 2 & 2 \end{array} \right) \end{array}$$

Uncrowding SVT

Uncrowding operator Lenart 2000; Buch 2002; Bandlow, Morse 2012; Patrias 2016; Reiner, Tenner, Yong 2018

- Identify the topmost row in T containing a multicell.
- Let x be the largest letter in that row which lies in a multicell.
- Delete x and perform RSK algorithm into the rows above. Repeat.
- Result is a single-valued skew tableau.

Example

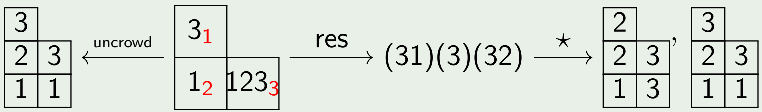


Connection to uncrowding map

Theorem (Morse, Pan, Poh, S. 2020)

Let $T \in \mathrm{SVT}^m(\lambda)$, $(\tilde{P}, \tilde{Q}) = \mathrm{uncrowd}(T)$, *and* $(P, Q) = \star \circ \mathrm{res}(T)$. *Then* $Q = \tilde{P}$.

Example



Hecke insertion (Buch 2008, Patrias, Pylyavskyy 2016)

Insert x to row R of an increasing tableau

- Try to append x to the right of R (record and terminate)
- Try to bump the smallest letter that is bigger (proceed to the next row)

$$\mathcal{H}^m \longleftrightarrow (P, Q)$$

Example

$$\mathbf{h} = (2)(31)(\,)(32) = \begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{3} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{3} & \mathbf{2} \end{bmatrix}.$$

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array} \to \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \to \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 2 \\ \hline\end{array} = P,$$

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \to \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \to \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{13} \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 13 \\ \hline\end{array} = Q.$$

Hecke insertion and the residue map

Theorem (Morse, Pan, Poh, S. 2020)

Let $T \in \mathrm{SVT}(\lambda)$ and $[\mathbf{k}, \mathbf{h}]^t = \mathrm{res}(T)$. Apply Hecke row insertion from the right on $[\mathbf{k}, \mathbf{h}]^t$ to obtain the pair of tableaux (P, Q). Then $Q = T$.

Example

$$T = \begin{array}{|c|c|} \hline 2_{1} & 4_{2} \\ \hline 1_{2} & 23_{3} \\ \hline \end{array} \xrightarrow{\text{res}} (2)(3)(31)(2) = \begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & \mathbf{3} & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

$$\begin{array}{|c|} \hline 2 \\ \hline \end{array} \to \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} \to \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array} \to \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array} \to \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} = P.$$

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \to \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} \to \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 2 \\ \hline \end{array} \to \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 23 \\ \hline \end{array} \to \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 23 \\ \hline \end{array} = Q.$$

Future Work

- Crystal structure for the non-321 avoiding case (beyond skew shapes)
- Demazure crystal structure to compute the intersection number?

Thank you !