Conference: Integrable systems & quantum groups

Crystal bases in statistical mechanics, representation theory and combinatorics

Lecture 1: Crystal bases
Applications to symmetric fcts

Lecture 2: Virtual crystals
Promotion
Cyclic sieving phenomenon

Lecture 3: Diagram algebras, insertion algorithms, plethysm

Lecture 2

Anne Schilling, UC Davis

- Virtual crystals
- Promotion
- Cyclic sieving phenomenon

based on work with

- Okado, Shimozono (~2003)
- Fourier, Shimozono (~2007)
- Pappe, Pfannerer, Simone (2022)
  arXiv: 2212.13588

Motivation

- Invariant subspaces $\mathrm{Inv}(V_1 \otimes \dots \otimes V_N)$
- $\dim \mathrm{Inv}(V_1 \otimes \dots \otimes V_N)$
  $=$ # highest weight elements of weight 0
  in $B_1 \otimes \dots \otimes B_N$ $=: \dim \mathrm{Inv}(B_1 \otimes \dots \otimes B_N)$
- Symmetric group S_N acts on $V_1 \otimes \dots \otimes V_N$
  by permuting tensor positions
- Action of long cycle on $\mathrm{Inv}(V_1 \otimes \dots \otimes V_N)$
  corresponds to promotion on $\mathrm{Inv}(B_1 \otimes \dots \otimes B_N)$
  Westbury 2016

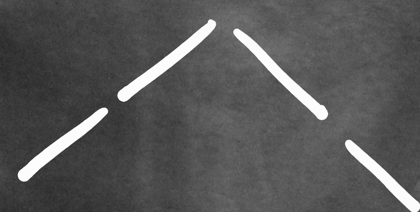
- $\mathrm{Inv}(B_1 \otimes \ldots \otimes B_N)$, promotion and q-deformation $\sum_{b \in \mathrm{Inv}(B_1 \otimes \ldots \otimes B_N)} q^{E(b)}$ gives cyclic sieving phenomenon

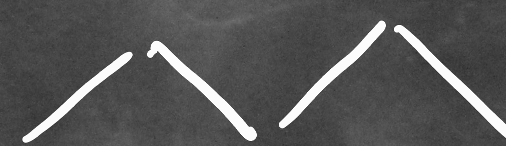
$\mathrm{Inv}(B^{\otimes N})$ type A_1

$\mathrm{Inv}(B^{\otimes N})$ = highest weight elements in $B^{\otimes N}$ of weight zero

Example $\boxed{1} \xrightarrow{1} \boxed{2}$

$N=4$ $\quad 2\otimes 2\otimes 1\otimes 1 \qquad 2\otimes 1\otimes 2\otimes 1$





Dyck paths of length N

$\mathrm{Inv}(\mathcal{B}_{\square}^{\otimes N})$ type C_r

$$\mathcal{B}_{\square} \qquad \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{\bar{r}} \xrightarrow{r-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

Example

		oscillating tableaux
$\mathrm{Inv}(\mathcal{B}_{\square}^{\otimes 2})$:	$\boxed{\bar{1}} \otimes \boxed{1}$	$\emptyset, \square, \emptyset$
$\mathrm{Inv}(\mathcal{B}_{\square}^{\otimes 4})$:	$\boxed{\bar{1}} \otimes \boxed{\bar{2}} \otimes \boxed{2} \otimes \boxed{1}$	$\emptyset, \square, \begin{smallmatrix}\square\\\square\end{smallmatrix}, \square, \emptyset$
	$\boxed{\bar{1}} \otimes \boxed{\bar{1}} \otimes \boxed{1} \otimes \boxed{1}$	$\emptyset, \square, \square\square, \square, \emptyset$
	$\boxed{\bar{1}} \otimes \boxed{1} \otimes \boxed{\bar{1}} \otimes \boxed{1}$	$\emptyset, \square, \emptyset, \square, \emptyset$

$\mathrm{Inv}(B_{spin}^{\otimes N})$ type B_r

B_{spin} $\quad \substack{+\\+} \xrightarrow{2} \substack{-\\+} \xrightarrow{1} \substack{+\\-} \xrightarrow{2} \substack{-\\-}$ $\quad$ type B_2

Example

2-fans of Dyck paths

$\mathrm{Inv}(B_{spin}^{\otimes 2}) \quad \substack{-\\-} \otimes \substack{+\\+} \; :$

$\mathrm{Inv}(B_{spin}^{\otimes 4}) \quad \substack{-\\-} \otimes \substack{-\\-} \otimes \substack{+\\+} \otimes \substack{+\\+}$

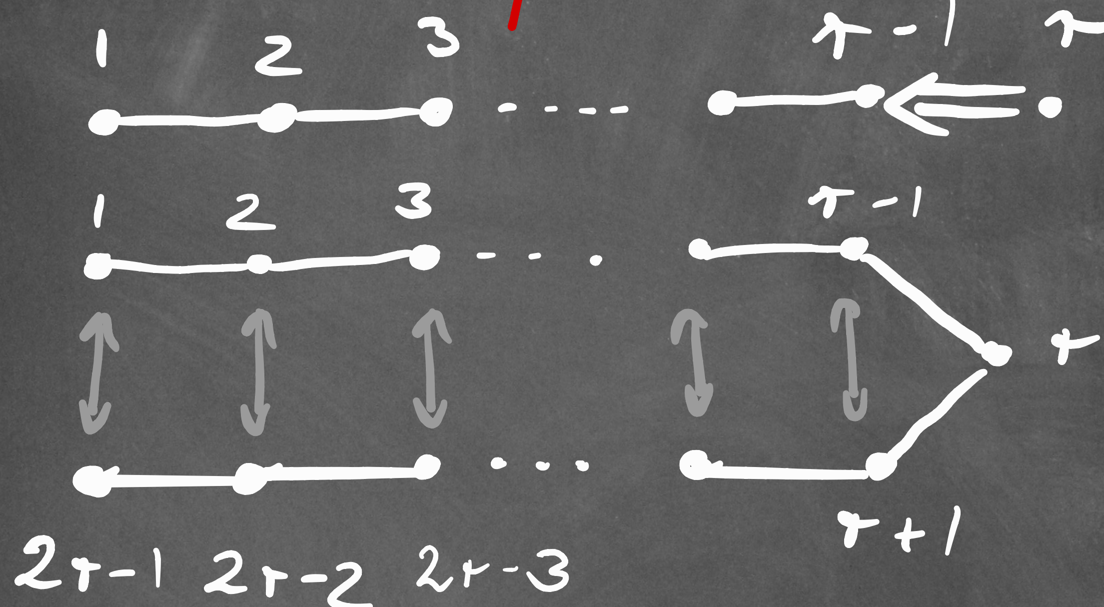
$\substack{-\\-} \otimes \substack{+\\+} \otimes \substack{-\\-} \otimes \substack{+\\+}$

$\substack{-\\-} \otimes \substack{+\\-} \otimes \substack{-\\+} \otimes \substack{+\\+}$

Virtual crystals

Embedding of algebras $X \hookrightarrow Y$

Typical example



$$C_r \hookrightarrow A_{2r-1}$$

Dynkin diagrams

$\sigma(i) = \{i, 2r-i\}$ $\quad 1 \le i < r$ $\qquad \gamma_i = 1$ $\quad 1 \le i < r$

$\sigma(r) = \{r\}$ $\qquad \gamma_r = 2$

Embedding of root and weight lattice:

$$\omega_i^X \mapsto \gamma_i \sum_{j \in \sigma(i)} \omega_j^Y$$

$$\alpha_i^X \mapsto \gamma_i \sum_{j \in \sigma(i)} \alpha_j^Y$$

$\hat{V}$ crystal of type Y with crystal operators $\hat{e}_i, \hat{f}_i$ (ambient crystal)

Virtual crystal operators

$$e_i = \prod_{j \in \sigma(i)} \hat{e}_j^{\gamma_i}$$

$$f_i = \prod_{j \in \sigma(i)} \hat{f}_j^{\gamma_i}$$

Definition A *virtual crystal* $V \subseteq \hat{V}$ is a subset s.t.

(V1) $\hat{V}$ is a crystal associated to representation

(V2) $\hat{\varepsilon}_j(b) = \hat{\varepsilon}_{j'}(b) \quad \forall j, j' \in \sigma(i)$

$\hat{\varphi}_j(b) = \hat{\varphi}_{j'}(b)$

aligned

Both are multiples of γ_i

Define $\varepsilon_i(b) = \frac{1}{\gamma_i} \hat{\varepsilon}_j(b)$

$\varphi_i(b) = \frac{1}{\gamma_i} \hat{\varphi}_j(b)$

$\forall b \in V$

$i \in I^\times$

$j \in \sigma(i)$

(V3) $V \cup \{\emptyset\} \subseteq \hat{V} \cup \{\emptyset\}$ is closed under e_i, f_i

and
$$\varepsilon_i(b) = \max\{k \mid e_i^{k}(b) \neq \emptyset\}$$
$$\varphi_i(b) = \max\{k \mid f_i^{k}(b) \neq \emptyset\}$$

Example

$\hat{V} = B_{\square} \otimes B_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$ type A_3

$f_1 = \hat{f}_1 \hat{f}_3$

$f_2 = \hat{f}_2^{\,2}$

$$B_{\square} \qquad \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}} \qquad \text{type } C_2$$

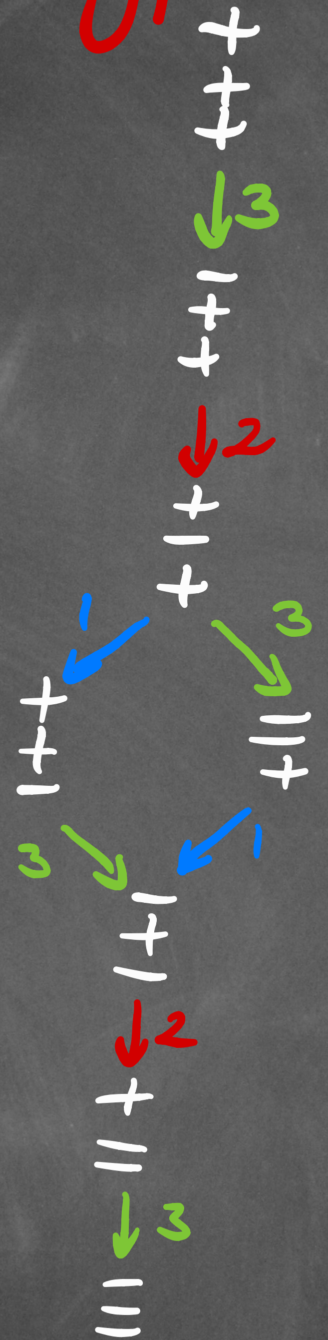
$$\hat{V} \supseteq V \qquad \boxed{1} \otimes \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \xrightarrow{1,3} \boxed{2} \otimes \begin{array}{|c|}\hline 4\\\hline 2\\\hline 1\\\hline\end{array} \xrightarrow{2,2} \boxed{3} \otimes \begin{array}{|c|}\hline 4\\\hline 3\\\hline 1\\\hline\end{array} \xrightarrow{1,3} \boxed{4} \otimes \begin{array}{|c|}\hline 4\\\hline 3\\\hline 2\\\hline\end{array}$$

Theorem $V \subseteq \hat{V}$, $W \subseteq \hat{W}$ virtual crystals

$\Rightarrow V \otimes W \subseteq \hat{V} \otimes \hat{W}$ virtual crystal

B_{spin} type B_r

B_3

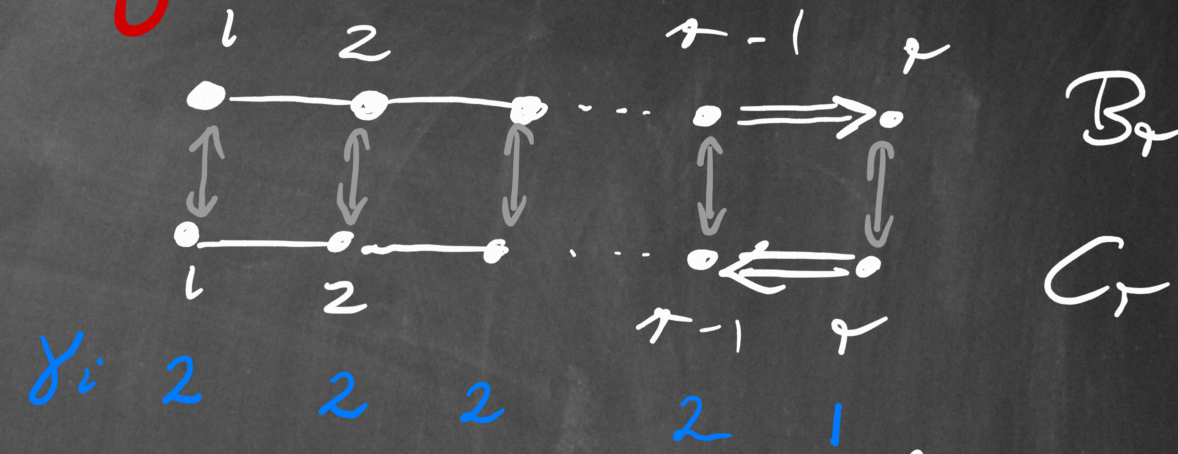


f_i changes $\begin{array}{cl} \vdots & \\ - & i+1 \\ + & i \\ \vdots & \end{array} \rightarrow \begin{array}{cl} \vdots & \\ + & i+1 \\ - & i \\ \vdots & \end{array}$

f_r change $\begin{array}{cl} + & r \\ \vdots & \end{array} \rightarrow \begin{array}{cl} - & r \\ \vdots & \end{array}$

B_{spin} type B_r as virtual crystal

$\hat{V} = B_{\square}^{\otimes r}$ of type C_r



V = component of $\boxed{r} \otimes \boxed{r-1} \otimes \cdots \otimes \boxed{1} \in \hat{V}$

$= \{ v_r \otimes v_{r-1} \otimes \cdots \otimes v_1 \mid v_i > v_j,\ |v_i| \neq |v_j|\ i \neq j \}$

under the order $1 < 2 < \cdots < r < \bar{r} < \cdots < \bar{2} < \bar{1}$

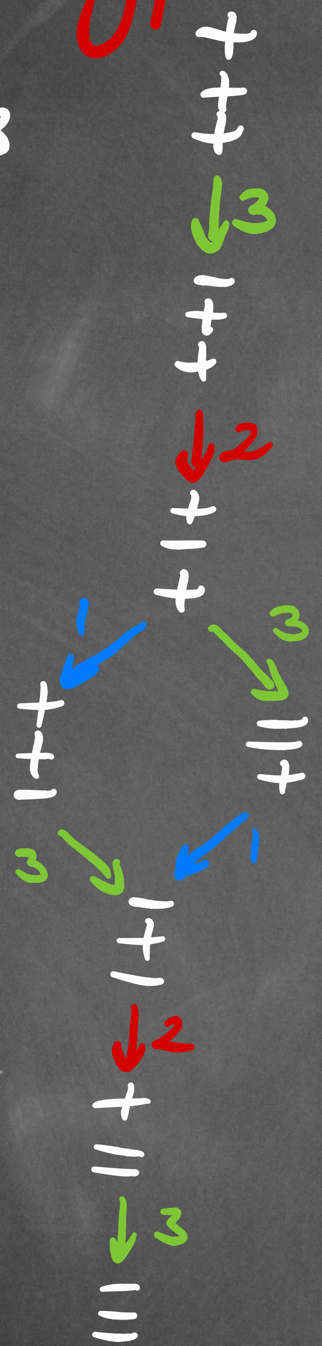
$|i| = |\bar{i}| = i$

$f_i = \hat{f}_i^{\,2},\ e_i = \hat{e}_i^{\,2} \quad 1 \le i < r$

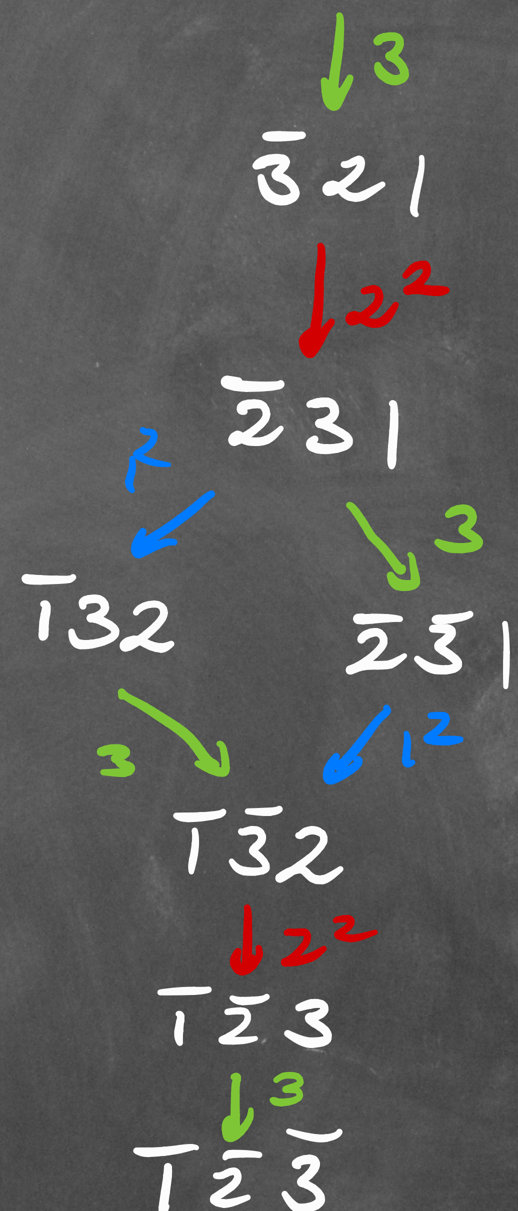
$f_r = \hat{f}_r,\ e_r = \hat{e}_r$

Bspin type B_r

type B_3



$321 \in B_{\square}^{\otimes 3}$



type C_3

$f_1 = \hat{f}_1^{\,2}$

$f_2 = \hat{f}_2^{\,2}$

$f_3 = \hat{f}_3$

Promotion

Crystal commutor (Henriques, Kamnitzer 2006)

$$\sigma : B_\lambda \otimes B_\mu \to B_\mu \otimes B_\lambda$$
$$b \otimes c \mapsto \eta(\eta(c) \otimes \eta(b))$$

Lusztig involution

$$\eta : B_\lambda \to B_\lambda$$

η maps highest weight to lowest weight

maps f_i to $e_{i'}$ with $w_0(\alpha_i) = -\alpha_{i'}$

long element

Definition

C crystal, $u \in C^{\otimes n}$ highest weight element

Then

$$\mathrm{pr}(u) = \delta_{C^{\otimes n-1}, C}(u)$$

Example

$\underbrace{221}_{B_{\square}^{\otimes 3}}\underbrace{1}_{B_{\square}} \in B_{\square}^{\otimes 4}$ type A_1

$\eta(221) = 121$

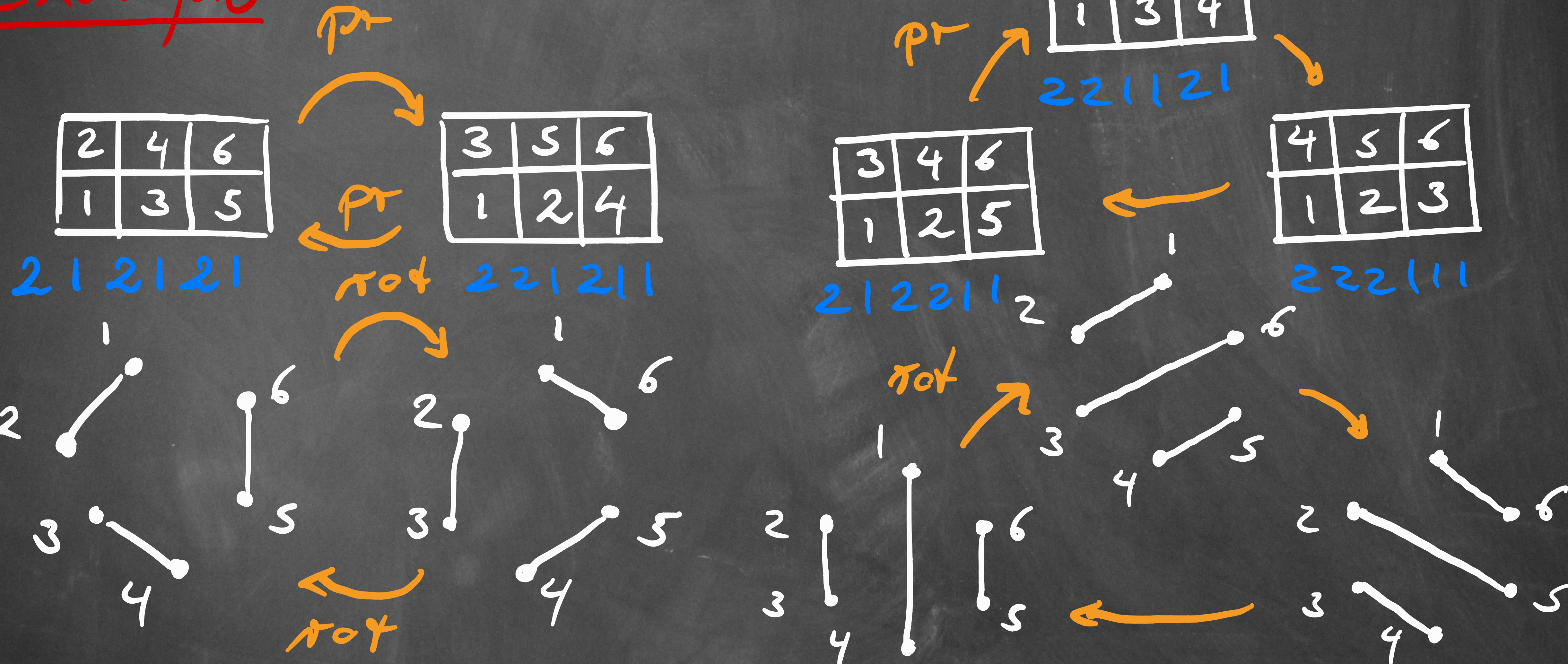
$\eta(1) = 2$

$\Rightarrow \mathrm{pr}(2211) = \underbrace{2}_{B_{\square}}\underbrace{121}_{B_{\square}^{\otimes 3}}$

Promotion and rotation

$\mathrm{cd}: \mathrm{Inv}(B^{\otimes n}) \to$ chord diagrams

Example



Theorem

$$\mathrm{cd} \circ \mathrm{pr} = \mathrm{rot} \circ \mathrm{cd}$$

A_{n-1}, C_n Pfannerer, Rubey, Westbury 2020

adjoint vector

vector

Peterson, Pylyavskyy, Rhoades 2009

Patrias 2019

Kuperberg web's 1996

B_n Pappe, Pfannerer, S., Simone 2022

spin, vector

Outline of construction

$\Psi : \mathrm{Inv}(\mathcal{B}^{\otimes N}) \to$ chord diagrams

constructed in two ways:

- fillings of promotion-evacuation diagrams
  Lenart 2008 → pr and rot intertwine
- Fomin growth diagrams Fomin 1986
  Krattenthaler 2006
  → injectivity

using virtual crystals

Example:

$$\begin{array}{ccccccccccccccc} - & & - & & + & & - & & + & & - & & + & & + \\ - & \otimes & - & \otimes & - & \otimes & + & \otimes & + & \otimes & - & \otimes & + & \otimes & + \\ - & & - & & - & & - & & + & & + & & + & & + \end{array}$$

type B_3

(1) We apply promotion a total of $n = 8$ times, to obtain the full orbit.

$$\begin{array}{ccccccccccccccccc}
000 & 111 & 222 & 311 & 422 & 331 & 222 & 111 & 000 & & & & & & & & \\
 & 000 & 111 & 200 & 311 & 220 & 111 & 000 & 111 & 000 & & & & & & & \\
 & & 000 & 111 & 222 & 311 & 220 & 111 & 222 & 111 & 000 & & & & & & \\
 & & & 000 & 111 & 200 & 111 & 200 & 311 & 200 & 111 & 000 & & & & & \\
 & & & & 000 & 111 & 220 & 311 & 422 & 311 & 222 & 111 & 000 & & & & \\
 & & & & & 000 & 111 & 220 & 331 & 220 & 311 & 200 & 111 & 000 & & & \\
 & & & & & & 000 & 111 & 222 & 111 & 220 & 111 & 220 & 111 & 000 & & \\
 & & & & & & & 000 & 111 & 000 & 111 & 200 & 311 & 220 & 111 & 000 & \\
 & & & & & & & & 000 & 111 & 222 & 311 & 422 & 331 & 222 & 111 & 000
\end{array}.$$

Example

(1) We apply promotion a total of $n = 8$ times, to obtain the full orbit.

000 111 222 311 422 331 222 111 000
000 111 200 311 220 111 000 111 000
000 111 222 311 220 111 222 111 000
000 111 200 111 200 311 200 111 000
000 111 220 311 422 311 222 111 000
000 111 220 331 220 311 200 111 000
000 111 222 111 220 111 220 111 000
000 111 000 111 200 311 220 111 000
000 111 222 311 422 331 222 111 000 .

(2) We group the results into the promotion matrix and fill the cells of the square grid according to Φ. For better readability we omitted zeros.

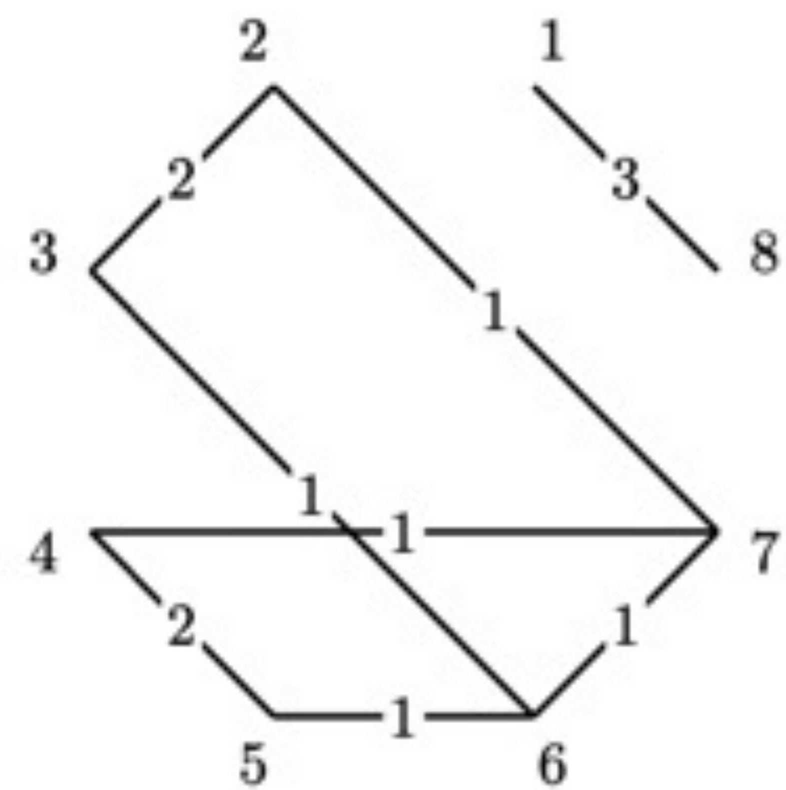
000		111		222		311		422		331		222		111		000
															3	
111		000		111		200		311		220		111		000		111
					2								**1**			
222		111		000		111		222		311		220		111		222
			2								**1**					
311		200		111		000		111		200		111		200		311
									2				**1**			
422		311		222		111		000		111		220		311		422
							2				**1**					
331		220		311		200		111		000		111		220		331
					1				**1**				**1**			
222		111		220		111		220		111		000		111		222
			1				**1**				**1**					
111		000		111		200		311		220		111		000		111
	3															
000		111		222		311		422		331		222		111		000

(2) We group the results into the promotion matrix and fill the cells of the square grid according to Φ. For better readability we omitted zeros.

000		111		222		311		422		331		222		111		000
															3	
111		000		111		200		311		220		111		000		111
					2								**1**			
222		111		000		111		222		311		220		111		222
			2								**1**					
311		200		111		000		111		200		111		200		311
									2				**1**			
422		311		222		111		000		111		220		311		422
							2				**1**					
331		220		311		200		111		000		111		220		331
					1				**1**				**1**			
222		111		220		111		220		111		000		111		222
			1				**1**				**1**					
111		000		111		200		311		220		111		000		111
	3															
000		111		222		311		422		331		222		111		000

(3) Regard the filling as the adjacency matrix of a graph, the chord diagram.

$$\mathsf{M}_F(\mathsf{F}) = \begin{pmatrix} 0&0&0&0&0&0&0&3\\ 0&0&2&0&0&0&1&0\\ 0&2&0&0&0&1&0&0\\ 0&0&0&0&2&0&1&0\\ 0&0&0&2&0&1&0&0\\ 0&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&0\\ 3&0&0&0&0&0&0&0 \end{pmatrix}$$



Cyclic sieving phenomenon

introduced by Reiner, Stanton, White 2004
as generalization of $q=-1$ phenomenon by
Stembridge.

Def X finite set
$C = \langle c \rangle$ cyclic group acting on X
ζ is $|C|^{th}$ root of unity
$f(q) \in \mathbb{Z}[q]$

Then (X, C, f) exhibits the cyclic sieving phenomenon
if $|X^{c^d}| = f(\zeta^d)$ $\forall d \geq 0$

fixed pt set under c^d

Cyclic sieving phenomenon for

- oscillating tableaux
- r-fans of Dyck paths

using the promotion action.

Polynomial $f(q)$ requires the energy function.

Local energy: $\mathcal{B}$ affine crystal

$$H : \mathcal{B} \otimes \mathcal{B} \to \mathbb{Z}$$

$$H(e_i(b_1 \otimes b_2)) = H(b_1 \otimes b_2) + \begin{cases} +1 & i=0 \\ & \varepsilon_0(b_1) > \varphi_0(b_2) \\ -1 & i=0 \\ & \varepsilon_0(b_1) \leq \varphi_0(b_2) \\ 0 & \text{else} \end{cases}$$

Example

$C_{\square}^{af}$ type $C_r^{(1)}$

$C_2^{(1)}$

$$\begin{tikzcd}
\boxed{1} \arrow[d, "1"] \\
\boxed{2} \arrow[d, "2"] \\
\boxed{\bar{2}} \arrow[d, "1"] \\
\boxed{\bar{1}} \arrow[uuu, bend left, "0"]
\end{tikzcd}$$

order

$1 < 2 < \cdots < r < \bar{r} < \cdots < \bar{1}$

$$H(a \otimes b) = \begin{cases} 0 & \text{if } a \leq b \\ 1 & \text{if } a > b \end{cases}$$

Example

B^{af}_{spin} type $B_r^{(1)}$

$B_2^{(1)}$

$$\begin{tikzcd}
\begin{smallmatrix}+\\+\end{smallmatrix} \arrow[d, "2"] \\
\begin{smallmatrix}-\\+\end{smallmatrix} \arrow[d, "1"] \\
\begin{smallmatrix}+\\-\end{smallmatrix} \arrow[d, "2"] \\
\begin{smallmatrix}-\\-\end{smallmatrix} \arrow[uuu, bend left, "0"]
\end{tikzcd}$$

\# of − signs in ε_i

$$H\left(\begin{matrix}\varepsilon_r\\ \vdots\\ \varepsilon_2\\ \varepsilon_1\end{matrix} \otimes \begin{matrix}+\\ \vdots\\ +\\ +\end{matrix}\right) = \left\lfloor \frac{m\begin{pmatrix}\varepsilon_r\\ \vdots \\ \varepsilon_1\end{pmatrix}+1}{2}\right\rfloor$$

Energy function

$$E : \mathcal{B}^{\otimes N} \to \mathbb{Z}$$

$$E(b_1 \otimes \cdots \otimes b_N) = \sum_{i=1}^{N-1} i\, H(b_i \otimes b_{i+1})$$

$\rightsquigarrow$ analogue of major index

Polynomial

$$f_{n,r}(q) = q^{*} \sum_{\substack{b \in \mathcal{B}^{\otimes 2n} \\ wt(b)=0 \\ e_i(b)=\emptyset \quad 1 \le i \le r}} q^{E(b)}$$

Theorem [PPSS 2022]

X set of highest weight elements of weight zero in $\mathcal{B}^{\otimes 2n}$, $\mathcal{B}$ minuscule

C_{2n} cyclic group of order $2n$ given by action of promotion on $\mathcal{B}^{\otimes 2n}$

$\Rightarrow (X, C_{2n}, f_{n,r}(q))$ exhibits cyclic sieving phenomenon

Fontaine, Kamnitzer 2014
Fourier, Littelmann 2007
Fourier, S., Shimozono 2007
Westbury 2016

Conjecture (see also Hopkins 2020)

In type B_r $(X, C_{2n}, g_{n,r}(q))$ exhibits the cyclic sieving phenomenon with

$$g_{n,r}(q) = \prod_{1 \le i \le j \le n-1} \frac{[i+j+2r]_q}{[i+j]_q}$$

$$[m]_q = 1 + q + q^2 + \cdots + q^{m-1}$$

q-deformation of # of r-fans of Dyck paths

Example

$$f_{3,2}(q) = q^{10} + q^{9} + 2q^{8} + q^{7} + 3q^{6} + q^{5} + 2q^{4} + q^{3} + q^{2} + 1$$

$$g_{3,2}(q) = q^{12} + q^{10} + q^{9} + 2q^{8} + q^{7} + 2q^{6} + q^{5} + 2q^{4} + q^{3} + q^{2} + 1$$

$$\Rightarrow g_{3,2}(q) = f_{3,2}(q) \mod (q^{6} - 1)$$

Thank
you!