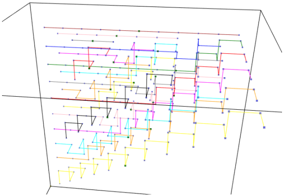
Lecture 3: Diagram algebras, insertion algorithms, and plethysm

Anne Schilling

Department of Mathematics, UC Davis

based on joint work with Rosa Orellana (Dartmouth), Franco Saliola (UQAM),
Mike Zabrocki (York), Algebraic Combinatorics (2022)
OSSZ, Laura Colmenarejo (NCSU) arXiv:2208.07258
COSSZ J. Algebra (2020)



Integrable systems and quantum groups
Osaka, Japan
March 8, 2023

Goal

- Exploration of variants of RSK
  - Insertion of multisets instead of integers
  - Enumerative manifestations of double centralizer theorem:

$$V = \bigoplus_{\lambda} V_\lambda = \bigoplus_{\lambda} U_\lambda \otimes W_\lambda \qquad \text{operators } \mathcal{A}, \mathcal{B} \text{ acting}$$

  $\mathcal{A}$ only acting on U_λ, $\mathcal{B}$ only acting on W_λ
- Applications to partition algebras
  - Insertion
    partition diagrams $\longrightarrow$ (standard tableau, multiset-valued tableau)
  - Well behaved with respect to subalgebras
  - dimensions of irreducibles = number of tableaux
- Uniform block permutation algebra $\to$ plethysm

Outline

The Robinson–Schensted–Knuth correspondence

- Robinson 1938: permutations in S_n
  $\longrightarrow \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$
- Schensted 1961: words of length n in $[k] = \{1, 2, \ldots, k\}$
  $\longrightarrow \bigcup_{\lambda \vdash n} \mathrm{SSYT}_{[k]}(\lambda) \times \mathrm{SYT}(\lambda)$
- Knuth 1970: generalized permutations over $[n]$ and $[k]$ of length ℓ
  $\longrightarrow \bigcup_{\lambda \vdash \ell} \mathrm{SSYT}_{[k]}(\lambda) \times \mathrm{SSYT}_{[n]}(\lambda)$

Generalized permutations

A, B ordered alphabets (i.e. $A = [n]$, $B = [k]$)

Definition

A generalized permutation is a two-line array $w = \begin{pmatrix} a_1 & a_2 & \cdots & a_\ell \\ b_1 & b_2 & \cdots & b_\ell \end{pmatrix}$ such that

- $a_1, \ldots, a_\ell \in A$, $b_1, \ldots, b_\ell \in B$
- $a_i \leqslant_A a_{i+1}$ for $1 \leqslant i \leqslant \ell - 1$
- $b_i \leqslant_B b_{i+1}$ whenever $a_i = a_{i+1}$

Example

Generalized permutation from [6] to [5]:

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 6 & 6 & 6 \\ 1 & 5 & 5 & 2 & 3 & 1 & 3 & 5 & 5 & 1 & 1 & 2 & 3 \end{pmatrix}$$

Row insertion

$$\begin{array}{|c|c|c|c|c|c|}
\hline
6 \\
\cline{1-1}
5 & 7 \\
\cline{1-2}
4 & 5 & 6 \\
\cline{1-3}
2 & 3 & 4 & 4 \\
\cline{1-4}
1 & 1 & 2 & 2 & 3 & 5 \\
\hline
\end{array} \qquad \longleftarrow \qquad 2$$

$$= \begin{array}{|c|c|c|c|c|c|}
\hline
6 & 7 \\
\cline{1-2}
5 & 5 \\
\cline{1-2}
4 & 4 & 6 \\
\cline{1-3}
2 & 3 & 3 & 4 \\
\cline{1-4}
1 & 1 & 2 & 2 & 2 & 5 \\
\hline
\end{array}$$

RSK correspondence

generalized permutation $w = \begin{pmatrix} a_1 & a_2 & \cdots & a_\ell \\ b_1 & b_2 & \cdots & b_\ell \end{pmatrix}$
Row insert $b_1, b_2, \ldots, b_\ell$ one by one
Record new box when inserting b_i by a_i

Theorem (Knuth 1970)

$\exists$ *bijection*

$$\text{generalized permutation from } A \text{ to } B \mapsto (P, Q)$$

- $\text{shape}(P) = \text{shape}(Q)$
- *P is semistandard tableau with entries in B*
- *Q is semistandard tableau with entries in A*

RSK and representation theory

Schensted 1961

- Combinatorial bijection

$$\{\text{words of length } n \text{ in } [k]\} \longrightarrow \bigcup_{\lambda \vdash n} \mathrm{SSYT}_{[k]}(\lambda) \times \mathrm{SYT}(\lambda)$$

- Enumerative result

$$k^n = \sum_{\lambda \vdash n} \#\mathrm{SSYT}_{[k]}(\lambda) \cdot \#\mathrm{SYT}(\lambda)$$

- Representation theory interpretation

  $GL_k \times S_n$-module $V^{\otimes n}$ where $V = \mathbb{C}^k$ (commuting actions)

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} W_k^\lambda \otimes S^\lambda$$

  W_k^λ is a simple left GL_k-module
  S^λ is a simple right S_n-module

Outline

Variant

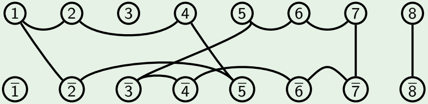
- Encoding of partition diagrams as generalized permutations with multisets
- RSK algorithm gives pairs of standard multiset tableaux
- Well behaved with respect to subalgebras
- Matches the representation theory and dimensions of Halverson and Jacobson (2018)
- New map from standard multiset tabelaux to Bratteli diagrams (different from Benkart and Halverson (2017))

Partition diagrams

Partition of two alphabets $[k]$ and $[\overline{k}]$

Example

$\pi = \{\{1,2,4,\overline{2},\overline{5}\},\{3\},\{5,6,7,\overline{3},\overline{4},\overline{6},\overline{7}\},\{8,\overline{8}\},\{\overline{1}\}\}$ represented by:

$\pi =$ 

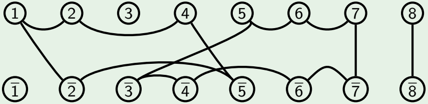
Partition algebra

$$P_k(n) = \mathrm{span}_{\mathbb{C}}\{\pi \mid \pi \vdash [k] \cup [\overline{k}]\}$$

(Non)propagating blocks

Example

$\pi = \{\{1,2,4,\overline{2},\overline{5}\},\{3\},\{5,6,7,\overline{3},\overline{4},\overline{6},\overline{7}\},\{8,\overline{8}\},\{\overline{1}\}\}$ represented by:

$\pi =$ 

A block is propagating if it contains vertices from both $[k]$ and $[\overline{k}]$.

Example

$\{1,2,4,\overline{2},\overline{5}\}$ is propagating.

Otherwise, the block is non-propagating.

Example

$\{3\}$ and $\{\overline{1}\}$ are non-propagating.

The correspondence

$\pi = \{\pi_1, \pi_2, \ldots, \pi_r\}$ set partition of $[k] \cup [\overline{k}]$
Order: last letter order

- $\pi_{j_1}, \pi_{j_2}, \ldots, \pi_{j_p}$ propagating blocks of π ordered as $\pi_{j_1}^{+} < \cdots < \pi_{j_p}^{+}$, where $\pi_j^{+} = \pi_j \cap [k]$ and $\pi_j^{-} = \pi_j \cap [\overline{k}]$
- $\sigma_{i_1}, \ldots, \sigma_{i_a} \subseteq [k]$ non-propagating blocks in $[k]$ ordered as $\sigma_{i_1} < \cdots < \sigma_{i_a}$
- $\tau_{i_1}, \ldots, \tau_{i_b} \subseteq [\overline{k}]$ non-propagating blocks in $[\overline{k}]$ ordered as $\tau_{i_1} < \cdots < \tau_{i_b}$

$$(P, Q) = RSK \begin{pmatrix} \pi_{j_1}^{+} & \pi_{j_2}^{+} & \cdots & \pi_{j_p}^{+} \\ \pi_{j_1}^{-} & \pi_{j_2}^{-} & \cdots & \pi_{j_p}^{-} \end{pmatrix}$$

$T = P$ by adjoining row containing $n - p - b$ empty cells followed by $\tau_{i_1}, \ldots, \tau_{i_b}$
$S = Q$ by adjoining row containing $n - p - a$ empty cells followed by $\sigma_{i_1}, \ldots, \sigma_{i_a}$

The correspondence – example

Example

$\pi = \{\{2,3,4,\bar{4},\bar{5}\},\{5,\bar{2},\bar{3}\},\{1,6,\bar{7},\bar{8}\},\{7,8\},\{9,\bar{6}\},\{\bar{1}\},\{\bar{9}\}\} \in P_9(18)$

$$\begin{pmatrix} \pi^+_{j_1} & \pi^+_{j_2} & \cdots & \pi^+_{j_p} \\ \pi^-_{j_1} & \pi^-_{j_2} & \cdots & \pi^-_{j_p} \end{pmatrix} = \begin{pmatrix} \{2,3,4\} & \{5\} & \{1,6\} & \{9\} \\ \{\bar{4},\bar{5}\} & \{\bar{2},\bar{3}\} & \{\bar{7},\bar{8}\} & \{\bar{6}\} \end{pmatrix}$$

Apply RSK:

$$P = \begin{array}{|c|c|} \hline \overline{45} & \overline{78} \\ \hline \overline{23} & \bar{6} \\ \hline \end{array} \qquad Q = \begin{array}{|c|c|} \hline 5 & 9 \\ \hline 234 & 16 \\ \hline \end{array}$$

Adjoin new rows:

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \overline{45} & \overline{78} \\ \cline{1-2} \overline{23} & \bar{6} \\ \hline & & & & & & & & & & & & \bar{1} & \bar{9} \\ \hline \end{array}$$

$$S = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 5 & 9 \\ \cline{1-2} 234 & 16 \\ \hline & & & & & & & & & & & & & 78 \\ \hline \end{array}$$

The correspondence – Theorem

$\mathrm{SMT}_{[k]}(\lambda)$ = set of standard multiset tableaux over alphabet $[k]$

Theorem (COSSZ'20)

Let $n \geqslant 2k$. $\exists$ bijection

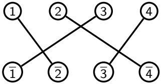
$$\Psi\colon \{\textit{set partitions of } [k] \cup [\overline{k}]\} \longrightarrow \bigcup_{\lambda \vdash n} \mathrm{SMT}_{[\overline{k}]}(\lambda) \times \mathrm{SMT}_{[k]}(\lambda)$$

Enumerative result

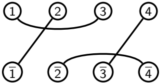
$$B(2k) = \sum_{\lambda \vdash n} \#\mathrm{SMT}_{[k]}(\lambda)^2$$

Restriction to subalgebras

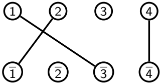
Subclasses of set partitions



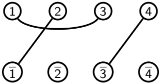
permutation



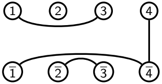
perfect matching



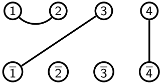
partial permutation



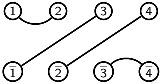
matching



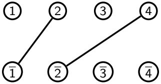
planar



planar matching



planar perfect matching



planar partial permutation

Subalgebras of the partition algebra $P_k(n)$

Subalgebra A_k	Diagrams spanning the subalgebra	Dimension
Partition algebra $P_k(n)$	all diagrams	$B(2k)$
Group algebra of symmetric group $\mathbb{C}S_k$	permutations	$k!$
Brauer algebra $B_k(n)$	perfect matchings	$(2k-1)!!$
Rook algebra $R_k(n)$	partial permutations	$\sum_{i=0}^{k} \binom{k}{i}^2 i!$
Rook-Brauer algebra $RB_k(n)$	matchings	$\sum_{i=0}^{k} \binom{2k}{2i} (2i-1)!!$
Temperley–Lieb algebra $TL_k(n)$	planar perfect matchings	$\frac{1}{k+1}\binom{2k}{k}$
Motzkin algebra $M_k(n)$	planar matchings	$\sum_{i=0}^{k} \frac{1}{i+1}\binom{2i}{i}\binom{2k}{2i}$
Planar rook algebra $PR_k(n)$	planar partial permutations	$\binom{2k}{k}$
Planar algebra $PP_k(n)$	planar diagrams	$\frac{1}{2k+1}\binom{4k}{2k}$

Properties under Ψ

A_k subalgebra of partition algebra
$\mathrm{SMT}_{A_k}(\lambda)$ set of standard multiset-valued tableaux under Ψ for A_k

Definition

$T \in \mathrm{SMT}_{A_k}(\lambda)$

- T is matching if the first row contains sets of size less than or equal to 2 and all other rows contain only sets of size 1.
- Two sets S and S' are non-crossing if there do not exist elements $a, b \in S$ and $c, d \in S'$ such that $a < c < b < d$ or $c < a < d < b$.
- We say that $c \in [k]$ is between a set S if there exist $a, b \in S$ such that $a < c < b$.
- T is planar if
  - it has two rows
  - the sets in the first row are pairwise non-crossing
  - no element belonging to one of the sets in the second row is between any set in the tableau

Tableaux for subalgebras

Under the bijection Ψ, the tableaux are characterized as follows:

		properties characterizing SMT_{A_k}	
A_k	diagrams spanning A_k	sizes of entries in first row	other properties
$P_k(n)$	all diagrams	—	—
$PP_k(n)$	planar diagrams	—	planar
$\mathbb{C}S_k$	permutations	0	matching
$B_k(n)$	perfect matchings	0, 2	matching
$R_k(n)$	partial permutations	0, 1	matching
$RB_k(n)$	matchings	0, 1, 2	matching
$TL_k(n)$	planar perfect matchings	0, 2	matching & planar
$M_k(n)$	planar matchings	0, 1, 2	matching & planar
$PR_k(n)$	planar partial permutations	0, 1	matching & planar

Tableaux for subalgebras

Corollary

Let $n \geqslant 2k$ and $\lambda \vdash n$. For each of the algebras A_k let $V_{A_k}^{\overline{\lambda}}$ be the irreducible A_k-representation indexed by $\overline{\lambda}$. Then

$$\dim\left(V_{A_k}^{\overline{\lambda}}\right) = \#\mathrm{SMT}_{A_k}(\lambda).$$

Corollary

If $n \geqslant 2k$, then for each subalgebra A_k of the partition algebra $P_k(n)$, we have

$$\dim(A_k) = \sum_{\lambda \vdash n} \left(\#\mathrm{SMT}_{A_k}(\lambda)\right)^2.$$

Diagram algebras

- Restrict diagonal action of GL_n on $V^{\otimes k}$ to $S_n \subseteq GL_n$: for $\sigma \in S_n$

$$\sigma(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = \sigma v_{i_1} \otimes \cdots \otimes \sigma v_{i_k}$$

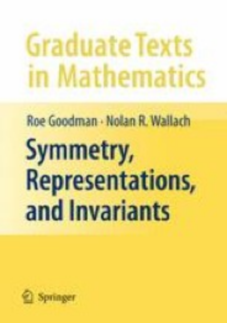
- What commutes with this action?
  Answer: Partition algebra $P_k(n)$ Martin, Jones 1990s
- Basis: set partitions of $\{1, 2, \ldots, k\} \cup \{\overline{1}, \overline{2}, \ldots, \overline{k}\}$

Remark

- S_k and GL_n form a centralizer pair
- $P_k(n)$ and S_n form a centralizer pair

Martin and Jones

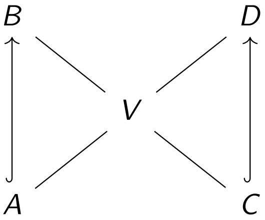
See-Saw pairs



(See book by Goodman, Wallach)

See-Saw pairs

$A \hookrightarrow B$ algebra embedding $\quad \mathrm{Res}_A^B \; V_B^{\lambda} = \bigoplus_{\mu} \left(V_A^{\mu}\right)^{\oplus c_{\lambda\mu}}$

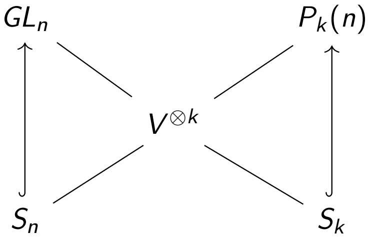


•B and C centralizer pair
•A and D centralizer pair

1. Indices for the simple modules for B and C are the same.
2. Indices for the simple modules for A and D are the same.

$$\mathrm{Res}_C^D \; V_D^{\mu} = \bigoplus_{\lambda} \left(V_C^{\lambda}\right)^{\oplus c_{\lambda\mu}}$$

Our See-Saw pair



$$\mathrm{Res}_{S_n}^{GL_n} \; V_{GL_n}^{\lambda} = \bigoplus_{\mu} \left(V_{S_n}^{\mu}\right)^{\oplus r_{\lambda\mu}}$$

$$\mathrm{Res}_{S_k}^{P_k(n)} \; V_{P_k(n)}^{\mu} = \bigoplus_{\lambda} \left(V_{S_k}^{\lambda}\right)^{\oplus r_{\lambda\mu}}$$

Idea: Restrict representations of $P_k(n)$ to S_k

The approach

$\mathcal{U}_k$ uniform block permutation algebra

$$\underbrace{S_k \hookrightarrow}_{\text{special cases of plethysm}} \mathcal{U}_k \underbrace{\hookrightarrow P_k(n)}_{\text{generalized LR coefficients}}$$

Goal: Combinatorial model for the representation theory of $\mathcal{U}_k$

Outline

Uniform block permutations

Tanabe 1997, Kosuda 2006

Party algebra, centralizer algebra for complex reflection groups

Definition

The set partition $d = \{d_1, d_2, \ldots, d_\ell\}$ of $[k] \cup [\bar{k}]$ is uniform if $|d_i \cap [k]| = |d_i \cap [\bar{k}]|$ for all $1 \leqslant i \leqslant \ell$. Let

$$\mathcal{U}_k = \left\{ d \vdash [k] \cup [\bar{k}] : d \text{ uniform} \right\}.$$

Example

$$d = \{\{2, \bar{4}\}, \{5, \bar{7}\}, \{1, 3, \bar{1}, \bar{2}\}, \{4, 6, \bar{3}, \bar{6}\}, \{7, 8, 9, \bar{5}, \bar{8}, \bar{9}\}\}$$

Think of d as a size-preserving bijection

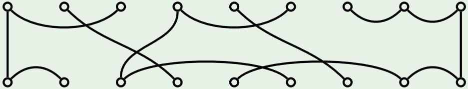
$$\begin{pmatrix} \{2\} & \{5\} & \{1,3\} & \{4,6\} & \{7,8,9\} \\ \{4\} & \{7\} & \{1,2\} & \{3,6\} & \{5,8,9\} \end{pmatrix}$$

$\Rightarrow$ Elements of $\mathcal{U}_k$ are called uniform block permutations

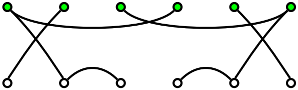
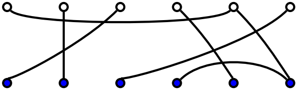
Uniform block permutations – continued

Example

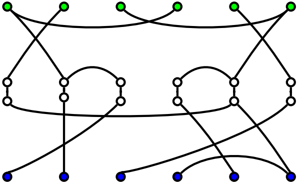
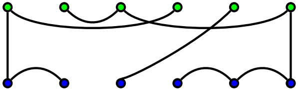
Diagram for $\{\{1,3,\bar{1},\bar{2}\},\{2,\bar{4}\},\{4,6,\bar{3},\bar{6}\},\{5,\bar{7}\},\{7,8,9,\bar{5},\bar{8},\bar{9}\}\}$



The product of

$d =$  and $d' =$ 

is obtained by stacking the diagrams of d and d':

$dd' =$  $=$ 

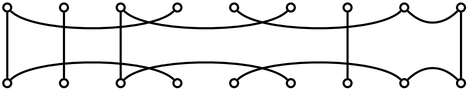
Idempotents

For every set partition π of $[k]$ we define:

$$e_\pi = \{A \cup \bar{A} : A \in \pi\} \in \mathcal{U}_k$$

where $\bar{A} = \{\bar{i} : i \in A\}$. For example,

$e_{2|7|14|36|589}$ =



Lemma

The set $E(\mathcal{U}_k) = \{e_\pi : \pi \vdash [k]\}$ *is a complete set of idempotents in* $\mathcal{U}_k$.

Maximal subgroups

Definition

M finite monoid, e idempotent

Maximal subgroup: $G_e =$ unique largest subgroup of M containing e

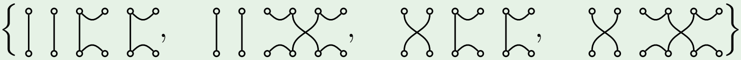
Lemma

The maximal subgroup of $\mathcal{U}_k$ at the idempotent e_π is

$$G_{e_\pi} = \{d \in \mathcal{U}_k : \mathrm{top}(d) = \mathrm{bot}(d) = \pi\}$$

Example

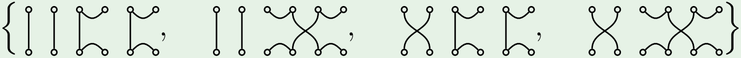
For $\pi = \{\{1\},\{2\},\{3,4\},\{5,6\}\}$

$G_{e_\pi} =$ 

Maximal subgroups – continued

Example

For $\pi = \{\{1\},\{2\},\{3,4\},\{5,6\}\}$ with $\text{type}(\pi) = (1^2 2^2)$

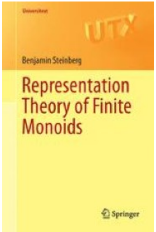
$G_{e_\pi} =$ 

Theorem

For $\pi \vdash [k]$ *with* $\text{type}(\pi) = (1^{a_1} 2^{a_2} \ldots k^{a_k})$

$$G_{e_\pi} \simeq S_{a_1} \times S_{a_2} \times \cdots \times S_{a_k}$$

Representation theory of $\mathcal{U}_k$



(See book by Steinberg 2016)

Indexing set of simple modules

$$I_k = \left\{ \left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}\right) : \lambda^{(i)} \text{ are partitions such that } \sum_{i=1}^{k} i|\lambda^{(i)}| = k \right\}$$

Example

$I_3 = \{((3), \emptyset, \emptyset), ((2,1), \emptyset, \emptyset), ((1,1,1), \emptyset, \emptyset), ((1), (1), \emptyset), (\emptyset, \emptyset, (1))\}$

Characters, symmetric functions, and plethysm

Theorem (OSSZ 2022)

Multiplicity of $V_{S_k}^{\mu}$ *in* $\mathrm{Res}_{S_k}^{\mathcal{U}_k} V_{\mathcal{U}_k}^{\vec{\lambda}}$ *is* $\left\langle s_{\lambda(1)}[s_1] s_{\lambda(2)}[s_2] \cdots s_{\lambda(k)}[s_k],\ s_{\mu} \right\rangle$

Outline

Plethysm via representations of GL_n

Definition

$GL_n(\mathbb{C})$ = invertible $n \times n$ matrices

- GL_n-representation $\rho\colon GL_n \to GL_m$
- GL_m-representation $\tau\colon GL_m \to GL_r$
- Composition is GL_n-representation

$$\tau \circ \rho\colon GL_n \to GL_r$$

Definition

Character of composition is plethysm:

$$\mathrm{char}(\tau \circ \rho) = \mathrm{char}(\tau)[\mathrm{char}(\rho)]$$

Frobenius map

R^n space of class functions of GL_n
Λ^n ring of symmetric functions of degree n

Power sum symmetric function p_λ

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}$$
$$p_r = x_1^r + x_2^r + \cdots$$

Schur function s_λ

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(T)}$$

Frobenius map – continued

Definition

The Frobenius characteristic map is $\mathrm{ch}^n \colon R^n \to \Lambda^n$

$$\mathrm{ch}^n(\chi) = \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_\mu p_\mu$$

where $z_\mu = 1^{a_1} a_1! 2^{a_2} a_2! \cdots$ for $\mu = 1^{a_1} 2^{a_2} \cdots$

Remark

The irreducible character χ^λ indexed by λ under the Frobenius map is

$$\mathrm{ch}^n(\chi^\lambda) = s_\lambda$$

by the identity

$$s_\lambda = \sum_{\mu} \frac{1}{z_\mu} \chi^\lambda_\mu p_\mu$$

Plethysm problem

Problem

Find a combinatorial interpretation for the coefficients $a^{\nu}_{\lambda\mu} \in \mathbb{N}$ *in the expansion*

$$s_\lambda[s_\mu] = \sum_{\nu} a^{\nu}_{\lambda\mu} s_\nu$$

Problem

Find a crystal on tableaux of tableaux which explains $a^{\nu}_{\lambda\mu}$.

Thank you !

Remark (Take away)

Plethysm is hard!

Remark (Take away)

Integrable systems, representation theory and combinatorics all play hand in hand!