q-deformations of the Tsetlin library

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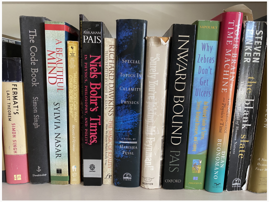
UC Berkeley Combinatorics seminar

Berkeley, February 11, 2026

based on joint with

Arvind Ayyer, Sarah Brauner, Jan de Gier

arXiv:2601.21195



Tsetlin library

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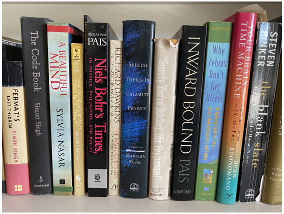

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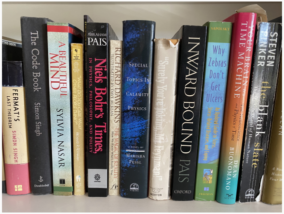

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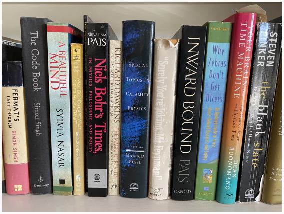
Tsetlin library



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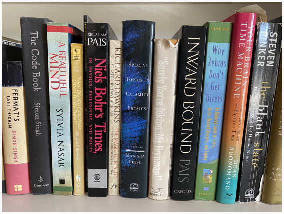

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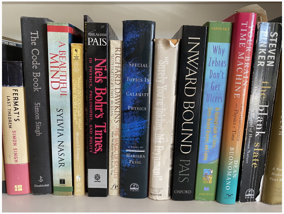

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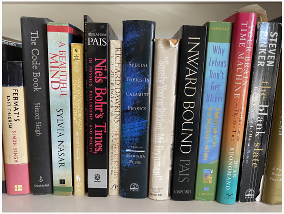

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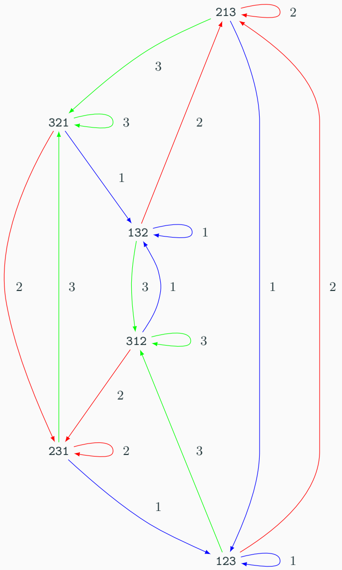
Popular books end up at the front!

Example: 3 books



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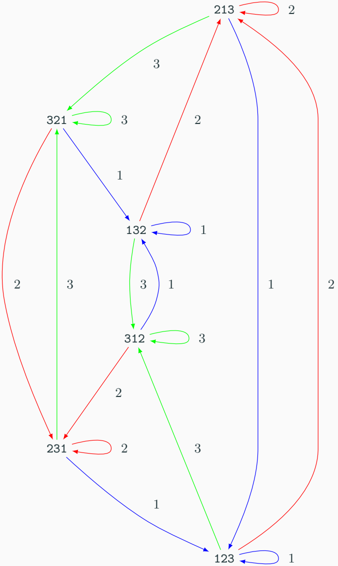

State space:

$\Omega = \{123, 132, 213, 231, 312, 321\}$

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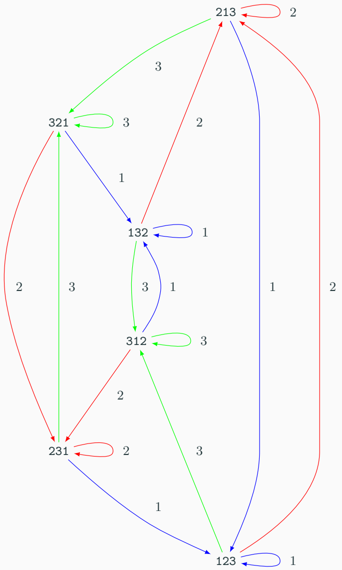

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Transition matrix:

$$\mathcal{T}(\mathsf{x}) = \begin{pmatrix} x_1 & 0 & x_2 & 0 & x_3 & 0 \\ 0 & x_1 & x_2 & 0 & x_3 & 0 \\ x_1 & 0 & x_2 & 0 & 0 & x_3 \\ x_1 & 0 & 0 & x_2 & 0 & x_3 \\ 0 & x_1 & 0 & x_2 & x_3 & 0 \\ 0 & x_1 & 0 & x_2 & 0 & x_3 \end{pmatrix}$$

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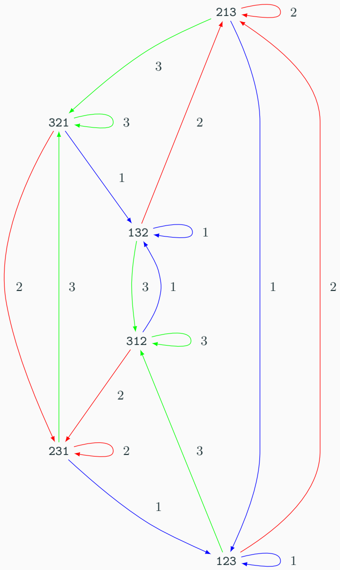

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Stationary distribution:

$\Psi \cdot \mathcal{T}(\mathsf{x}) = \Psi$

$$\Psi = \left(\frac{x_1 x_2}{x_2 + x_3}, \frac{x_1 x_3}{x_2 + x_3}, \frac{x_1 x_2}{x_1 + x_3}, \frac{x_2 x_3}{x_1 + x_3}, \frac{x_1 x_3}{x_1 + x_2}, \frac{x_2 x_3}{x_1 + x_2} \right)$$

Random to top shuffle Markov chain

Random to top generator

$$\mathcal{T}(\mathsf{x}) = \sum_{i=1}^{n} s_{i-1} \cdots s_1 \mathcal{X}(\mathsf{x})$$

such that

$$\pi \cdot \mathcal{T}(\mathsf{x}) = \sum_{i=1}^{n} x_{\pi_i} (\pi_i, \pi_1, \ldots, \hat{\pi}_i, \ldots, \pi_n)$$

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Stationary state (Hendricks '72)

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Alternative derivation of stationary state: **geometric finite semigroups** (Diaconis, Brown, Ayyer, Steinberg, Thiéry, Rhodes, S., . . .)

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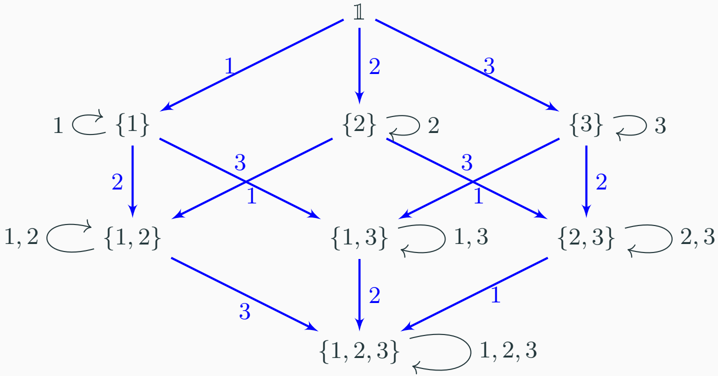
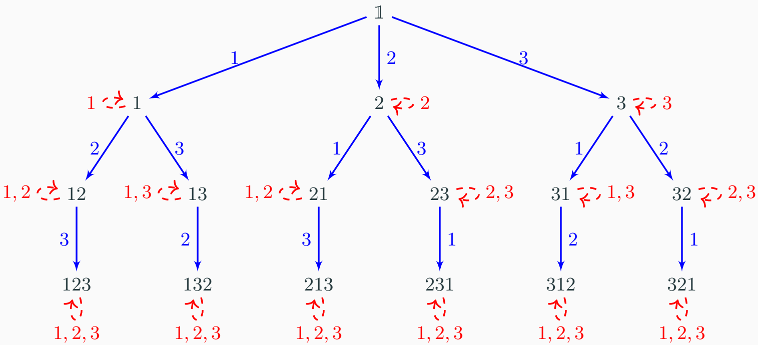
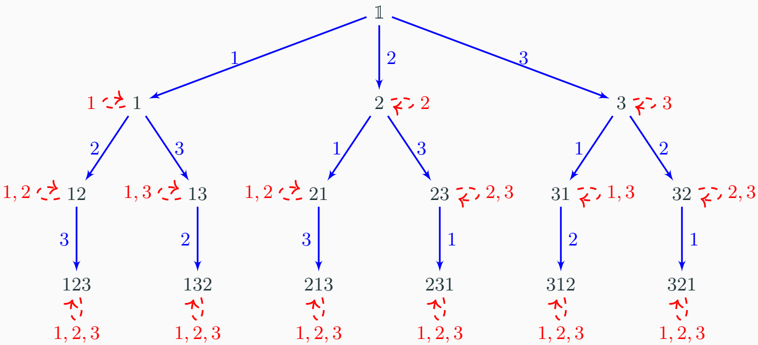
Right Cayley graph $\mathsf{Cay}(P(n), [n])$



Tsetlin library: Karnofsky–Rhodes expansion





Minimal ideal: $K =$ permutations of n letters

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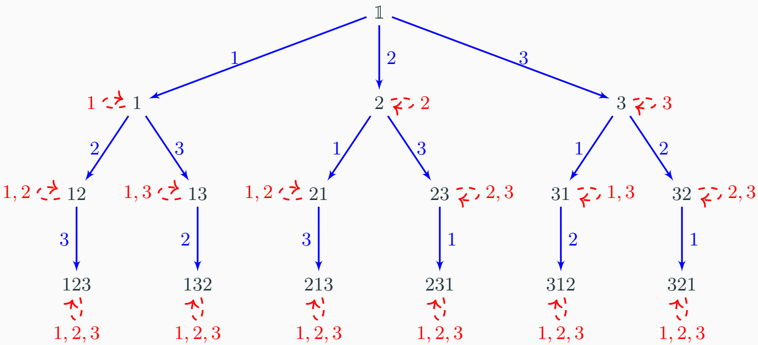

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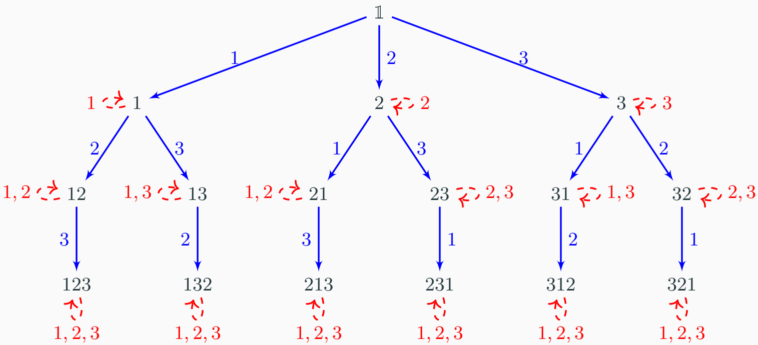

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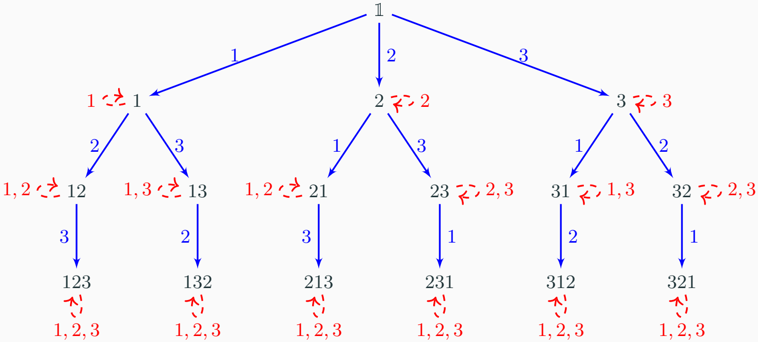
Remark (Take away)

Left versus right is important!

Tsetlin library: Karnofsky–Rhodes expansion



Property: unique simple path to permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$

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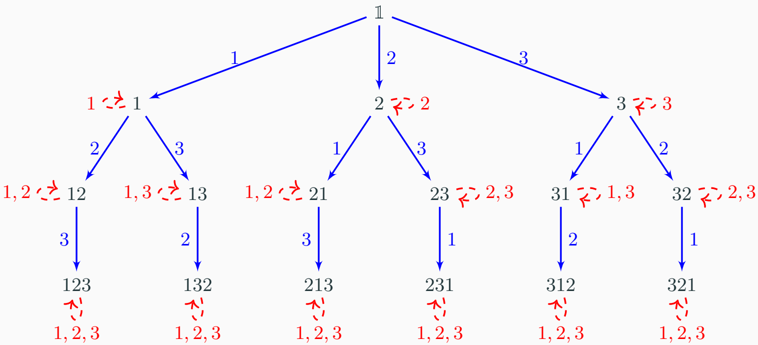

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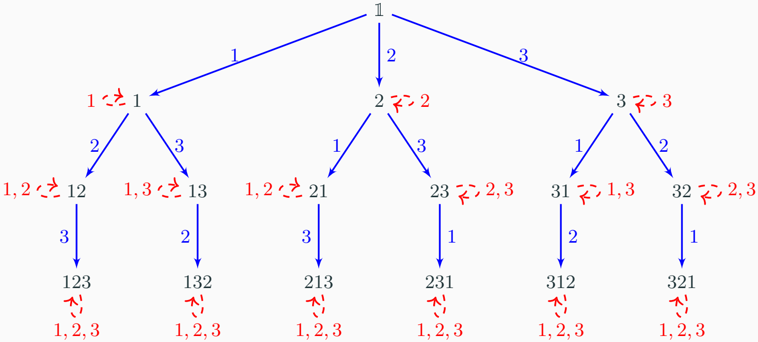
All paths to π: $\pi_1 \pi_1^\star \pi_2 \{\pi_1, \pi_2\}^\star \cdots \pi_i \{\pi_1, \dots, \pi_i\}^\star \cdots \pi_n$



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Remark (Intuition, see Brown and Diaconis 1998)

Stationary distribution Ψ_s for $s \in K$

= probability of starting at $\mathbb{1}$ and reaching s

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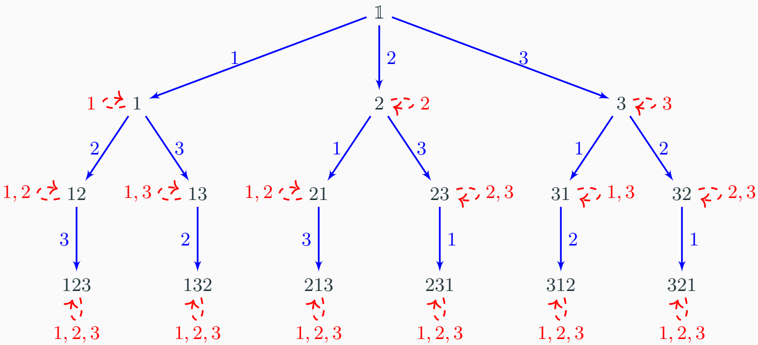

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Extended to all finite Markov chains with Rhodes in 2019

Tsetlin library: stationary distribution

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Geometric series: $$\sum_{s \in a^\star} x_s = \sum_{\ell=0}^{\infty} x_a^\ell = \frac{1}{1 - x_a} \quad \text{where} \quad x_s = \prod_{i \in s} x_i$$

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Theorem

Stationary distribution for Tsetlin library

$$\Psi_\pi = \sum_{s \in \pi_1 \pi_1^\star \pi_2 \{\pi_1, \pi_2\}^\star \cdots \pi_n} x_s = \prod_{i=1}^{n} \frac{x_{\pi_i}}{1 - \sum_{j=1}^{i-1} x_{\pi_j}}$$

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Rederivation of result of Hendricks 1972.

Eigenvalues

Phatarfod 1991

$d_n := n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ = number of derangements of $\mathfrak{S}_n$

Theorem

For every subset $S = \{i_1, \ldots, i_k\} \subseteq [n]$, there is an eigenvalue $x_{i_1} + \cdots + x_{i_k}$ which occurs with multiplicity d_{n-k}.

q-deformations

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Recall: Tsetlin library as action of symmetric group $\mathfrak{S}_n$ on permutations:

$$\mathcal{T}(\mathsf{x}) = \sum_{i=1}^{n} s_{i-1} \cdots s_1 \mathcal{X}(\mathsf{x})$$

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Hecke algebra $\mathcal{H}_n(q)$:

$$\begin{aligned} T_i T_j &= T_j T_i && \text{if } |i-j| > 1 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leqslant i \leqslant n-2 \\ (T_i + 1)(T_i - q) &= 0 && \text{for } 1 \leqslant i \leqslant n-1 \end{aligned}$$

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The q-analogue of the random-to-top shuffle becomes
(Lusztig 2006 and Axelrod-Freed, Brauner, Chiang, Commins, Lang 2024)

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Question: How to introduce probabilities $\mathcal{X}(\mathsf{x})$?

q-deformations

We define three q-deformations of the Tsetlin library:

$$\mathcal{T}(q, \mathsf{x}) = \sum_{i=1}^{n} T_{i-1} \cdots T_1 \, \mathcal{X}(\mathsf{x})$$

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- Only $\mathcal{T}_{G/B}(q, \mathsf{x})$ is related to a left-regular band.
- We obtain $\mathcal{T}_{\mathfrak{S}_n}(q, \mathsf{x})$ and $\mathcal{T}_{W_{\mathsf{m}}}(q, \mathsf{x})$ by lumping.

Results

- Definition of the Markov chains $\mathcal{T}_{\mathfrak{S}_n}(q, \mathsf{x})$, $\mathcal{T}_{W_{\mathsf{m}}}(q, \mathsf{x})$, $\mathcal{T}_{G/B}(q, \mathsf{x})$

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- Eigenvalues and multiplicities

q-Tsetlin library on permutations

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Action of Hecke generator T_i on permutations:

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Definition (ABGS2026)

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with

$$\pi \cdot \mathcal{X}(q, \mathsf{x}) = \frac{x_{\pi_1}}{q^{n-\pi_1}} \pi, \qquad \sum_{i=1}^{n} x_i = 1.$$

q-Tsetlin library on permutations

Various specializations

q-Tsetlin library $\qquad\qquad$ q-random-to-top

$$\begin{tikzcd}
\mathcal{T}(q, \mathsf{x}) \arrow[rrr, "x_i = q^{n-i}/[n]_q"] \arrow[d, "q=1"'] & & & \mathcal{T}(q) \arrow[d, "q=1"] \\
\mathcal{T}(\mathsf{x}) \arrow[rrr, "x_i = 1/n"'] & & & \mathcal{T}
\end{tikzcd}$$

Tsetlin library $\qquad\qquad$ random-to-top

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q-Tsetlin library　　　　　　　　q-random-to-top

$$\begin{tikzcd}
\mathcal{T}(q,\mathsf{x}) \arrow[r, "x_i = q^{n-i}/[n]_q"] \arrow[d, "q=1"'] & \mathcal{T}(q) \arrow[d, "q=1"] \\
\mathcal{T}(\mathsf{x}) \arrow[r, "x_i = 1/n"'] & \mathcal{T}
\end{tikzcd}$$

Tsetlin library　　　　　　　　random-to-top

q-integers:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}$$

Example

Example (q-Tsetlin library for $n = 3$)

$$\mathcal{T}(q, \mathsf{x}) = \begin{pmatrix} \frac{(q^2-q+1)x_1}{q^2} & \frac{(q-1)x_1}{q^2} & x_2 & 0 & x_3 & 0 \\ \frac{(q-1)x_1}{q} & \frac{x_1}{q} & x_2 & 0 & x_3 & 0 \\ x_1 & 0 & \frac{x_2}{q} & \frac{(q-1)x_2}{q} & 0 & x_3 \\ x_1 & 0 & 0 & x_2 & 0 & x_3 \\ 0 & x_1 & 0 & x_2 & x_3 & 0 \\ 0 & x_1 & 0 & x_2 & 0 & x_3 \end{pmatrix}$$

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Stationary distribution: left eigenvector

$$\Psi(q, \mathsf{x}) \cdot \mathcal{T}(q, \mathsf{x}) = (x_1 + x_2 + \cdots + x_n)\Psi(q, \mathsf{x})$$

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Example ($n = 3$ continued)

$$\Psi(q,\mathsf{x}) = \Bigg(qx_1 \frac{x_1 - q(x_1+x_2)}{x_1 - q^2}, x_1 \frac{x_1 - qx_1 - q^2 x_3}{x_1 - q^2}, \frac{qx_1x_2}{-x_2 + q}, \\ x_2 \frac{x_2 - q(x_2+x_3)}{x_2 - q}, \frac{x_1x_3}{x_1+x_2}, \frac{x_2x_3}{x_1+x_2}\Bigg)$$

Example

Example (q-Tsetlin library for $n = 3$ continued)

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Eigenvalues of $\mathcal{T}(q, \mathsf{x})$: $0, 0, q^{-2}x_1, q^{-1}x_2, x_3, x_1 + x_2 + x_3$

Stationary distribution

$$\kappa(\mathsf{b};\mathsf{x}) := \sum_{i=1}^{k} x_{b_i} q^{i+b_i-k-1} \quad \text{for } \mathsf{b} = (b_1, \ldots, b_k) \text{ weakly decreasing}$$

$$\kappa(\mathsf{b};\mathsf{x}) := \kappa(\mathsf{b}_>;\mathsf{x}) \quad \text{for } \mathsf{b}_> \text{ reordering of } \mathsf{b} \text{ in weakly decreasing order}$$

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Definition

- *Left-to-right minimum of $\pi \in \mathfrak{S}_n$ is π_j such that $\pi_i > \pi_j$ for all $1 \leqslant i < j$.*
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Theorem (ABGS2026)

$$\Psi(q,\mathsf{x})_\pi = \frac{q^{-\operatorname{inv}(\pi)}}{\prod_{k=1}^{n-1}\left(1 - q^{k-n-1}\kappa(\pi_1,\ldots,\pi_{k-1};\mathsf{x})\right)} \times \left(\prod_{\substack{k=1 \\ k\in\mathrm{LRM}(\pi)}}^{n-1} \kappa(\pi_k;\mathsf{x}) \prod_{\substack{k=1 \\ k\notin\mathrm{LRM}(\pi)}}^{n-1} \left(\kappa(\pi_{p_k},\ldots,\pi_k;\mathsf{x}) - q^{-1}\kappa(\pi_{p_k},\ldots,\pi_{k-1};\mathsf{x})\right) \right)$$

Eigenvalues

Theorem (ABGS2026)

For every $S = \{i_1 > i_2 > \cdots > i_k\} \subseteq [n]$*, there is an eigenvalue of* $\mathcal{T}(q, \mathsf{x})$

$$\lambda_S(q, \mathsf{x}) = \sum_{j=1}^{k} \frac{x_{i_j}}{q^{n-i_j-j+1}}$$

which occurs with multiplicity d_{n-k}*.*

q-Tsetlin library on words

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Action of Hecke algebra

$$w \cdot T_i := \begin{cases} q(w_1 \ldots w_{i+1}\ w_i \ldots w_n) & \text{if } w_{i+1} \leqslant w_i \\ (w_1 \ldots w_{i+1} w_i \ldots w_n) + (q-1)w & \text{if } w_{i+1} > w_i \end{cases}$$

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q-Tsetlin library on words

$$\mathcal{T}_{W_{\mathsf{m}}}(q, \mathsf{x}) := \sum_{i=1}^{n} T_{i-1} \cdots T_1\, \mathcal{X}_{W_{\mathsf{m}}}(q, \mathsf{x})$$

q-Tsetlin library on flags

Flags

The semigroup formed by union of sets is replaced by an action on flags over finite fields.

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The standard flag is given by

$$E_\bullet = (0, \langle e_1 \rangle, \langle e_1, e_2 \rangle, \ldots, \langle e_1, e_2, \ldots, e_n \rangle)$$

Number of flags

The number of lines in $\mathbb{F}_q^n$ is

$$[n]_q = \frac{1 - q^n}{1 - q}$$

The number of flags in $\mathbb{F}_q^n$ is $[n]_q!$

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Example ($n = 2$ and $q = 2$)

$\mathbb{F}_2^2$ has $[2]_2 = 3$ lines:

$$\langle e_1 \rangle,\ \langle e_2 \rangle \text{ and } \langle e_1 + e_2 \rangle$$

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There are $[2]_2! = 3$ flags in $\mathbb{F}_2^2$:

$$\begin{matrix} \{0\} \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \\ \{0\} \subseteq \langle e_1 + e_2 \rangle \subseteq \langle e_1, e_2 \rangle \end{matrix} \qquad \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \ (\alpha \in \mathbb{F}_2)$$

$$\{0\} \subseteq \langle e_2 \rangle \subseteq \langle e_1, e_2 \rangle \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Equivalence classes

Because of the action of $GL_n(\mathbb{F}_q)$, flags can be represented by equivalence classes $[\pi] = [(0, \langle v_1 \rangle, \langle v_1, v_2 \rangle, \ldots, \langle v_1, v_2, \ldots, v_n \rangle)]$ labeled by permutations.

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Example ($GL_3(\mathbb{F}_q)$)

$$[123] = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{pmatrix} \middle| \alpha, \beta, \gamma \in \mathbb{F}_q \right\}$$

$$[213] = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \alpha & \beta & 1 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{F}_q \right\}, \quad [132] = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 0 & 1 \\ \beta & 1 & 0 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{F}_q \right\}$$

$$[312] = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & \alpha & 1 \\ 1 & 0 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{F}_q \right\}, \quad [231] = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \alpha & 1 & 0 \end{pmatrix} \middle| \alpha \in \mathbb{F}_q \right\}$$

$$[321] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Hecke action on flags

The action of the Hecke generator T_i on a flag is given by

$$(V_0 \subseteq \cdots \subseteq V_n)T_i = \sum_{W \neq V_i} (V_0 \subseteq \cdots \subseteq V_{i-1} \subseteq W \subseteq V_{i+1} \subseteq \cdots \subseteq V_n),$$

where the sum is over all subspaces $V_{i-1} \subseteq W \subseteq V_{i+1}$ such that $\dim(W) = i$ and $W \neq V_i$. (Ram–Halverson and Iwahori)

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Example

On the flag $\langle(0,0,1)\rangle \subseteq \langle(0,0,1),(0,1,0)\rangle \in [321]$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} T_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \sum_{t \neq 0} \begin{pmatrix} 0 & 0 & 1 \\ t & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\cong \sum_{\alpha \in \mathbb{F}_q} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \alpha & 1 & 0 \end{pmatrix} \in [231]$$

The last sum gives rise to q terms (or flags) related to the permutation 231.

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This corresponds to Hecke action $[321]T_1 = q[231]$.

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Example

Another example for a flag of the form

$$\langle (0,1,\alpha) \rangle \subseteq \langle (0,1,\alpha),(1,0,\beta) \rangle \in [213]$$

$$\begin{aligned}
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \alpha & \beta & 1 \end{pmatrix} T_2 &\cong \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \alpha & 1 & 0 \end{pmatrix} + \sum_{t \neq 0} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \alpha & \beta + t & 1 \end{pmatrix} \\
&\cong \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \alpha & 1 & 0 \end{pmatrix} + \sum_{\substack{\gamma \in \mathbb{F}_q \\ \gamma \neq \beta}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \alpha & \gamma & 1 \end{pmatrix}.
\end{aligned}$$

This corresponds to $[213]T_2 = [231] + (q-1)[213]$.

q-Tsetlin library on flags is implemented by adding lines and variables associated to the topmost nonzero entry in line L_i

$$(V_0 \subseteq \cdots \subseteq V_n)\mathcal{T}_{G/B}(q, \mathsf{x}) = \sum_{j=1}^{[n]_q} \mathcal{X}(L_j)(V_0 \subseteq L_j \subseteq V_1 + L_j \subseteq \cdots \subseteq V_{n-1} + L_j \subseteq V_n)$$

where $\mathcal{X}(L_j) = \frac{x_i}{q^{n-i}}$ if $L_j = \langle e_i + \sum_{k=i+1}^{n} c_k e_k \rangle$.

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- The generators of the left regular band are the $[n]_q$ lines in $\mathbb{F}_q^n$.

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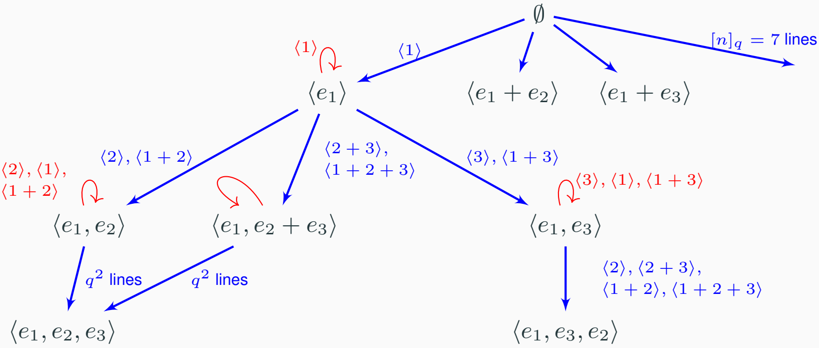
- The generators of the left regular band are the $[n]_q$ lines in $\mathbb{F}_q^n$.
- The line $L = \langle e_i + \sum_{k=i+1}^{n} c_k e_k \rangle$ has probability

$$y_i = \frac{x_i}{q^{n-i}}$$

because there are q^{n-i} lines L with leading term e_i since there are q choices for each coefficient c_k.

Example $n = 3$ and $q = 2$

Partial KR-expanded Cayley graph



$$F_\bullet = \langle e_1 \rangle \subseteq \langle e_1, e_3 \rangle \subseteq \langle e_1, e_3, e_2 \rangle$$

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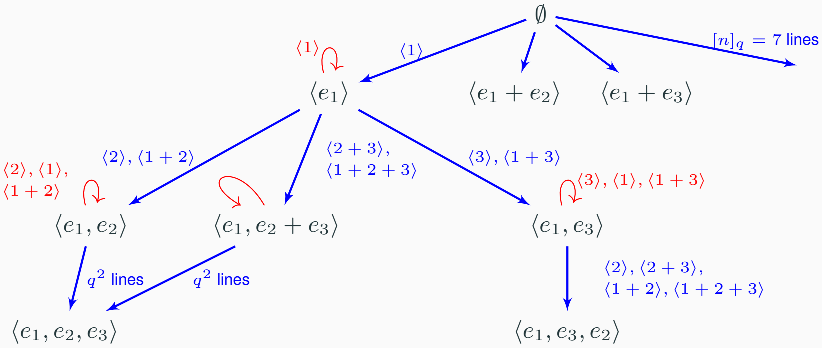

$$F_\bullet = \langle e_1 \rangle \subseteq \langle e_1, e_3 \rangle \subseteq \langle e_1, e_3, e_2 \rangle$$

$$\psi_{F_\bullet} = \frac{y_1}{1 - y_1} \times \frac{y_3 + (q-1)y_1}{1 - y_3 - qy_1} \times (qy_2 + q(q-1)y_1)$$

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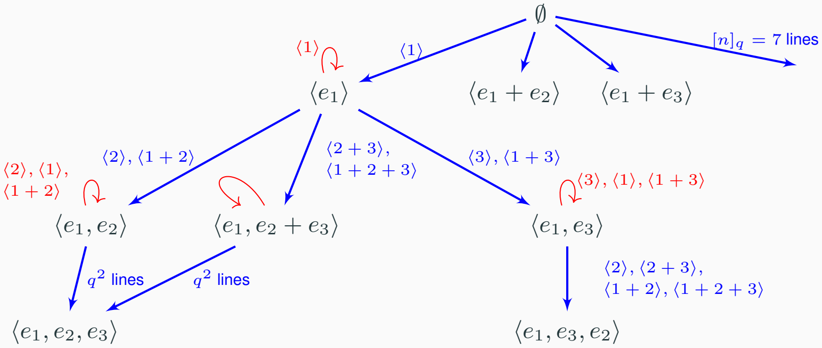

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$$= \frac{x_1}{q^2 - x_1} \frac{q^2 x_3 + (q-1)x_1}{q^2} = \frac{1}{q^2} \Psi_{[132]}$$

General stationary distribution

Fix permutation $\pi \in S_n$.

Define $b_k(s) = \#\{t \in \{\pi_1, \ldots, \pi_{k-1}\} \mid t > s\}$.

Define $\chi(s < \pi_k) = 1$ if $s < \pi_k$ and 0 else.

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Define $\chi(s < \pi_k) = 1$ if $s < \pi_k$ and 0 else.

Theorem (ABGS2026)

The stationary distribution for a flag associated to the permutation $\pi = \pi_1\pi_2 \ldots \pi_n$ for a general prime power q is given by

$$\psi_\pi = \frac{\prod_{k=1}^{n} f(\pi_1 \ldots \pi_k)}{\prod_{k=1}^{n-1}\left(1 - \sum_{s\in\{\pi_1,\ldots,\pi_k\}} \frac{x_s}{q^{n-s-b_{k+1}(s)}}\right)}$$

where

$$f(\pi_1 \ldots \pi_k) = \sum_{\substack{s\in\{\pi_1,\ldots,\pi_k\}\\ s\leqslant\pi_k}} \frac{x_s}{q^{n-s-b_k(s)}}(q-1)^{\chi(s<\pi_k)}.$$

Eigenvalues

q-derangement numbers (Wachs 1989)

$$d_n(q) = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k}{[k]_q!} q^{\binom{k}{2}}$$

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Theorem (ABGS2026)

Eigenvalues of $\mathcal{T}_{G/B}(x, q)$:

For every subset $S = \{i_1 > i_2 > \cdots > i_k\} \subseteq [n]$, there is an eigenvalue

$$\lambda_S(q, \mathsf{x}) = \sum_{j=1}^{k} \frac{x_{i_j}}{q^{n-i_j-j+1}}$$

Eigenvalues

q-derangement numbers (Wachs 1989)

$$d_n(q) = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k}{[k]_q!} q^{\binom{k}{2}}$$

Theorem (ABGS2026)

Eigenvalues of $\mathcal{T}_{G/B}(x, q)$:

For every subset $S = \{i_1 > i_2 > \cdots > i_k\} \subseteq [n]$, there is an eigenvalue

$$\lambda_S(q, \mathsf{x}) = \sum_{j=1}^{k} \frac{x_{i_j}}{q^{n-i_j-j+1}}$$

Multiplicities:

$\lambda_S(q, \mathsf{x})$ occurs with multiplicity $d_{n-k}(q) q^{(n-i_1)+(n-1-i_2)+\cdots+(n-k+1-i_k)}$ when $k \neq n$, and with multiplicity 1 when $k = n$.

Conclusion

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- Application to other Markov chains
  - multi-species ASEP
  - type C analogue
  - signed permutations
  - linear extension analogues