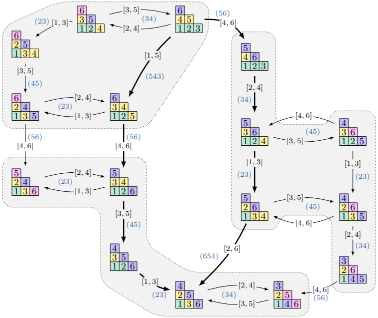
# Crystal skeletons: Combinatorics and axioms

Anne Schilling

Department of Mathematics, UC Davis



ICERM Workshop
Computation in Representation Theory
Brown University
November 10, 2025

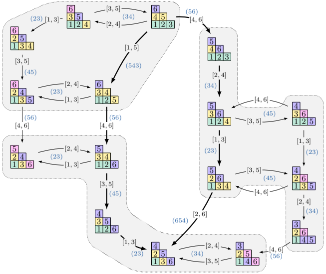
# Based on joint work with

Sarah Brauner, Sylvie Corteel, and Zajj Daugherty
preprint arXiv:2503.14782

# ~~Crystal skeletons: Combinatorics and axioms~~
# Crystals and their contractions

Anne Schilling

Department of Mathematics, UC Davis



ICERM Workshop
Computation in Representation Theory
Brown University
November 10, 2025

# Also based on joint work at ICERM with

Sarah Brauner, Zajj Daugherty, and Sarah Mason
preprint

# Thanks to stimulating research atmosphere at Mathematics Institutes

# Thanks to stimulating research atmosphere at Mathematics Institutes

- Banff January 2024 (Brauner, Corteel, Daugherty, S.)

# Thanks to stimulating research atmosphere at Mathematics Institutes

- Banff January 2024 (Brauner, Corteel, Daugherty, S.)
- IPAM April 2024 (Brauner, S.)

# Thanks to stimulating research atmosphere at Mathematics Institutes

- Banff January 2024 (Brauner, Corteel, Daugherty, S.)
- IPAM April 2024 (Brauner, S.)
- IAS June 2025 (Brauner, Corteel, Daugherty, S.)

# Thanks to stimulating research atmosphere at Mathematics Institutes

- Banff January 2024 (Brauner, Corteel, Daugherty, S.)
- IPAM April 2024 (Brauner, S.)
- IAS June 2025 (Brauner, Corteel, Daugherty, S.)
- ICERM September - now 2025 (Brauner, Daugherty, Mason, S.)

# Motivation

Crystal theoretic interpretation of the following identities:

# Motivation

Crystal theoretic interpretation of the following identities:

$$s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(T)}$$

- $s_\lambda$ Schur function indexed by partition $\lambda$
- $F_\alpha$ Gessel's quasisymmetric function indexed by composition $\alpha$

# Motivation

Crystal theoretic interpretation of the following identities:

$$s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(T)}$$

- $s_\lambda$ Schur function indexed by partition $\lambda$
- $F_\alpha$ Gessel's quasisymmetric function indexed by composition $\alpha$

$$s_\lambda = \sum_{\alpha \in S_\ell \cdot \lambda} \mathcal{YQS}_\alpha \quad \text{and} \quad \mathcal{YQS}_\alpha = \sum_\beta d_{\alpha,\beta} F_\beta$$

- $\mathcal{YQS}_\alpha$ Young quasisymmetric Schur function indexed by composition $\alpha$

# Motivation

## Tiling and contraction

$$
\begin{array}{ccccc}
\text{Crystal } B(\lambda) & & \text{Crystal skeleton } \mathrm{CS}(\lambda) & & \\
\mathrm{char} B(\lambda) = s_\lambda & & \mathrm{char}\mathrm{CS}(\lambda) = s_\lambda & & \\
\rotatebox{90}{$\subseteq$} & \longrightarrow & \rotatebox{90}{$\subseteq$} & \longrightarrow & \text{Bruhat order} \\
\text{Quasicrystal } Q_\beta & & \text{Quasicrystal skeleton } \mathrm{QCS}_\alpha & & \\
\mathrm{char} Q_\beta = F_\beta & & \mathrm{char}\mathrm{QCS}_\alpha = \mathcal{YQS}_\alpha & &
\end{array}
$$

# Motivation

Quasisymmetric expansions are known in various settings, where Schur expansions are still elusive:

# Motivation

Quasisymmetric expansions are known in various settings, where Schur expansions are still elusive:

- LLT polynomials

# Motivation

Quasisymmetric expansions are known in various settings, where Schur expansions are still elusive:

- LLT polynomials
- modified Macdonald polynomials

# Motivation

Quasisymmetric expansions are known in various settings, where Schur expansions are still elusive:

- LLT polynomials
- modified Macdonald polynomials
- plethysm of Schur functions

# Motivation

Quasisymmetric expansions are known in various settings, where Schur expansions are still elusive:

- LLT polynomials
- modified Macdonald polynomials
- plethysm of Schur functions
- characters of higher Lie modules (Thrall's problem)

# Motivation

Quasisymmetric expansions are known in various settings, where Schur expansions are still elusive:

- LLT polynomials
- modified Macdonald polynomials
- plethysm of Schur functions
- characters of higher Lie modules (Thrall's problem)

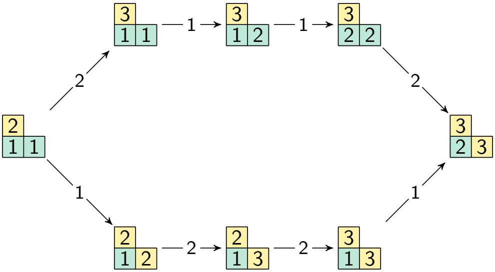
(Young) quasisymmetric Schur expansion implies Schur expansion

# Outline

# Crystals

Combinatorial representation theory of $\mathfrak{sl}_n$ (Kashiwara – Abel prize!!!)

Crystal: $B(2,1)$



# Crystals

- Vertices: semistandard Young tableaux $\mathrm{SSYT}(\lambda)$

# Crystals

- Vertices: semistandard Young tableaux $\mathrm{SSYT}(\lambda)$
- Edges: $f_i(T) = T'$
  - Row reading word $\mathrm{row}(T)$
  - Successively pair $i+1$ before $i$

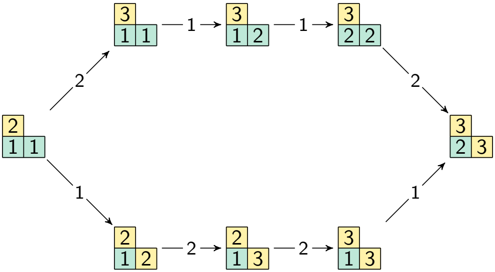
# Crystals

- Vertices: semistandard Young tableaux $\mathrm{SSYT}(\lambda)$
- Edges: $f_i(T) = T'$
  - Row reading word $\mathrm{row}(T)$
  - Successively pair $i+1$ before $i$
  - $$f_i(i^r(i+1)^s) = \begin{cases} i^{r-1}(i+1)^{s+1} & \text{if } r > 0 \\ 0 & \text{else} \end{cases}$$

# Crystals

Character of crystal $\mathrm{ch}B(\lambda) = s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(T)}$

Crystal: $B(2,1)$



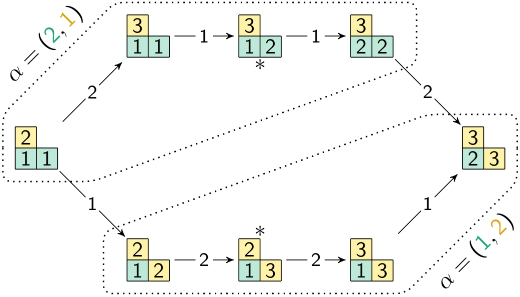

Schur function $s_{(2,1)}$

# Outline

# Quasicrystals

Group tableaux by standardization – labeled by descent composition

Crystal: $B(2,1)$



# Quasicrystals

2017 $\rightarrow$ Cain, Gray, Guilherme, Malheiro, Ribeiro, Rodrigues, Rodrigues
quasicrystals and hypoplactic monoids

2023 Maas-Gariépy

- Quasicrystal: $Q_T$ indexed by standard Young tableaux $T \in \mathrm{SYT}(\lambda)$
- Vertices:

$$Q_T = \{b \in B(\lambda) \mid \mathrm{std}(b) = T\}$$

# Quasicrystals

2017 $\rightarrow$ Cain, Gray, Guilherme, Malheiro, Ribeiro, Rodrigues, Rodrigues
quasicrystals and hypoplactic monoids

2023 Maas-Gariépy

- Quasicrystal: $Q_T$ indexed by standard Young tableaux $T \in \mathrm{SYT}(\lambda)$
- Vertices:

$$Q_T = \{b \in B(\lambda) \mid \mathrm{std}(b) = T\}$$

- Edges: $f_i(T) = T'$ in crystal is quasicrystal edge
  if and only if no $i+1$ is paired with $i$

# Quasicrystals

## Theorem

$$Q_T \cong Q_{T'} \quad \textit{if } \mathrm{Des}(T) = \mathrm{Des}(T')$$

# Quasicrystals

**Theorem**

$$Q_T \cong Q_{T'} \quad \textit{if } \mathrm{Des}(T) = \mathrm{Des}(T')$$

$\mathrm{SYT}(\lambda)$ set of standard Young tableaux of shape $\lambda$
$T \in \mathrm{SYT}(\lambda)$ (French notation), $\lambda \vdash n$

# Quasicrystals

## Theorem

$$Q_T \cong Q_{T'} \quad \textit{if}\ \mathrm{Des}(T) = \mathrm{Des}(T')$$

$\mathrm{SYT}(\lambda)$ set of standard Young tableaux of shape $\lambda$
$T \in \mathrm{SYT}(\lambda)$ (French notation), $\lambda \vdash n$

## Definition

$i$ is a descent in $T$ if $i+1$ is in higher row of the tableau
$d_1 < d_2 < \cdots < d_k$ descents in $T$

$$\mathrm{Des}(T) = (d_1, d_2 - d_1, \ldots, d_k - d_{k-1}, n - d_k)$$

# Quasicrystals

## Theorem

$$Q_T \cong Q_{T'} \quad \textit{if } \mathrm{Des}(T) = \mathrm{Des}(T')$$

SYT($\lambda$) set of standard Young tableaux of shape $\lambda$

$T \in \mathrm{SYT}(\lambda)$ (French notation), $\lambda \vdash n$

## Definition

$i$ is a descent in $T$ if $i+1$ is in higher row of the tableau

$d_1 < d_2 < \cdots < d_k$ descents in $T$

$$\mathrm{Des}(T) = (d_1, d_2 - d_1, \ldots, d_k - d_{k-1}, n - d_k)$$

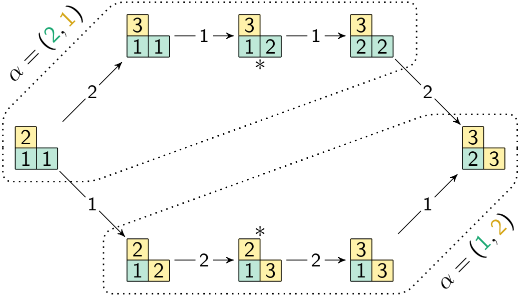
## Example

$$T = \begin{array}{|c|c|c|c|} \hline 7 \\ \cline{1-1} 3 & 5 \\ \cline{1-2} 1 & 2 & 4 & 6 \\ \hline \end{array} \qquad \text{descents } \{2,4,6\} \quad \mathrm{Des}(T) = (2,2,2,1)$$

# Quasicrystals

Character of quasicrystal: $\mathrm{ch}\, Q_T = F_{\mathrm{Des}(T)}$

Crystal: $B(2,1)$




Schur function $s_{(2,1)} = F_{(2,1)} + F_{(1,2)}$

# Gessel's quasisymmetric functions

Gessel's quasisymmetric functions (1984)

$$F_\alpha = \sum_{\substack{\beta \preccurlyeq \alpha \\ \text{refinement}}} M_\beta \qquad \text{with} \qquad M_\beta = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$$

# Gessel's quasisymmetric functions

Gessel's quasisymmetric functions (1984)

$$F_\alpha = \sum_{\substack{\beta \preccurlyeq \alpha \\ \text{refinement}}} M_\beta \qquad \text{with} \qquad M_\beta = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$$

## Example

$$\begin{aligned} F_{(1,2,1)} &= M_{(1,2,1)} + M_{(1,1,1,1)} \\ &= x_1 x_2^2 x_3 + x_1 x_2^2 x_4 + x_2 x_3^2 x_4 + \cdots \\ &\quad + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5 + \cdots \end{aligned}$$

# Schur functions

**Theorem (Gessel 1984)**

$$s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(T)}$$

# Schur functions

## Theorem (Gessel 1984)

$$s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(T)}$$

## Example

$$s_{(4,2)} = F_{(4,2)} + F_{(3,3)} + F_{(3,2,1)} + F_{(2,4)} + F_{(2,3,1)} + F_{(2,2,2)}$$

$$\begin{array}{|c|c|cc} \hline 5 & 6 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad \begin{array}{|c|c|cc} \hline 4 & 5 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{6} \\ \hline \end{array} \quad \begin{array}{|c|c|cc} \hline 4 & 6 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \quad \begin{array}{|c|c|cc} \hline 3 & 4 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{6} \\ \hline \end{array} \quad \begin{array}{|c|c|cc} \hline 3 & 6 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \quad \begin{array}{|c|c|cc} \hline 3 & 5 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{6} \\ \hline \end{array}$$

$$+ F_{(1,4,1)} + F_{(1,3,2)} + F_{(1,2,3)}$$

$$\begin{array}{|c|c|cc} \hline 2 & 6 & \multicolumn{2}{c}{} \\ \hline 1 & 3 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \quad \begin{array}{|c|c|cc} \hline 2 & 5 & \multicolumn{2}{c}{} \\ \hline 1 & 3 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{6} \\ \hline \end{array} \quad \begin{array}{|c|c|cc} \hline 2 & 4 & \multicolumn{2}{c}{} \\ \hline 1 & 3 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{6} \\ \hline \end{array}$$

# Outline

# Motivation

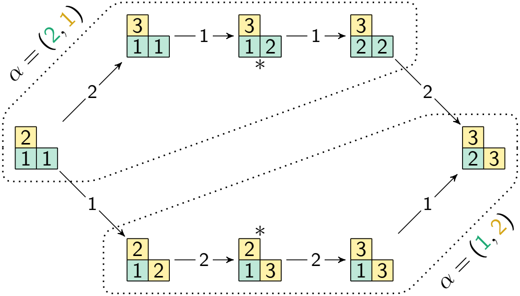
## Tiling and contraction

$$\begin{array}{ccccc} \text{Crystal } B(\lambda) & & \text{Crystal skeleton } \mathrm{CS}(\lambda) & & \\ \mathrm{char} B(\lambda) = s_\lambda & & \mathrm{char}\,\mathrm{CS}(\lambda) = s_\lambda & & \\ \cup\!\! \mid & \longrightarrow & \cup\!\! \mid & \longrightarrow & \text{Bruhat order} \\ \text{Quasicrystal } Q_\beta & & \text{Quasicrystal skeleton } \mathrm{QCS}_\alpha & & \\ \mathrm{char} Q_\beta = F_\beta & & \mathrm{char}\,\mathrm{QCS}_\alpha = \mathcal{YQS}_\alpha & & \end{array}$$

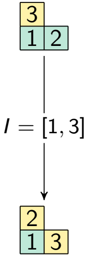
# Crystal skeletons

2023 Maas-Gariépy: Contract quasicrystals to a point

Crystal: $B(2,1)$




Crystal skeleton: $\mathrm{CS}(2,1)$




Schur function $s_{(2,1)} = F_{(2,1)} + F_{(1,2)}$

# Results: Combinatorics of crystal skeleton

2025 Brauner, Corteel, Daugherty, S.

Crystal skeleton $\mathrm{CS}(\lambda)$

# Results: Combinatorics of crystal skeleton

2025 Brauner, Corteel, Daugherty, S.

Crystal skeleton $\mathrm{CS}(\lambda)$

- **Vertices**
    1. Standard tableaux $T$
    2. Descent compositions $\alpha$

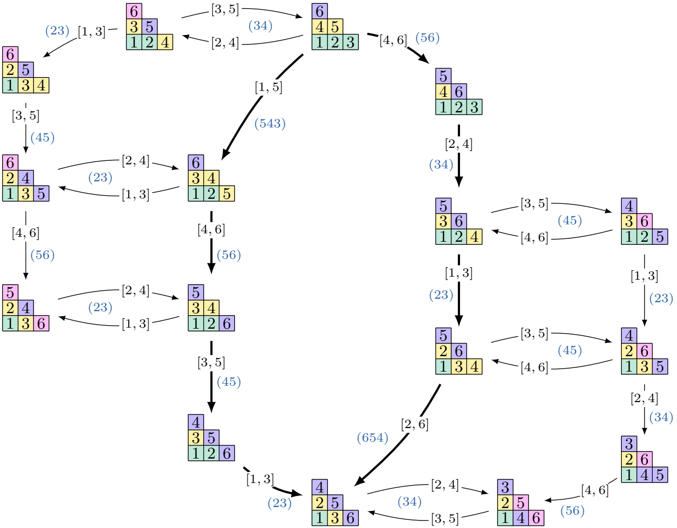
# Results: Combinatorics of crystal skeleton

2025 Brauner, Corteel, Daugherty, S.

Crystal skeleton $\mathrm{CS}(\lambda)$

- **Vertices**
  1. Standard tableaux $T$
  2. Descent compositions $\alpha$
- **Edges**
  1. Dyck pattern intervals: odd length intervals $I$ with $|I| \geqslant 3$
  2. Cycles

# Results: Combinatorics of crystal skeleton



# Edges: Dyck pattern intervals

## Definition

- $\pi \in S_n$ permutation
- $I = [i, i+2m] \subseteq [n]$ interval of length $2m+1 \geqslant 3$

The interval $I$ is a Dyck pattern interval of $\pi$ if

$$P(\pi|_I) = \begin{array}{|c|c|c|c|c}
\hline
i+m+1 & i+m+2 & \cdots & i+2m & \multicolumn{1}{c}{} \\
\hline
i & i+1 & \cdots & i+m-1 & \multicolumn{1}{c|}{i+m} \\
\hline
\end{array}$$

# Edges: Dyck pattern intervals

## Definition

- $\pi \in S_n$ permutation
- $I = [i, i+2m] \subseteq [n]$ interval of length $2m+1 \geqslant 3$

The interval $I$ is a Dyck pattern interval of $\pi$ if

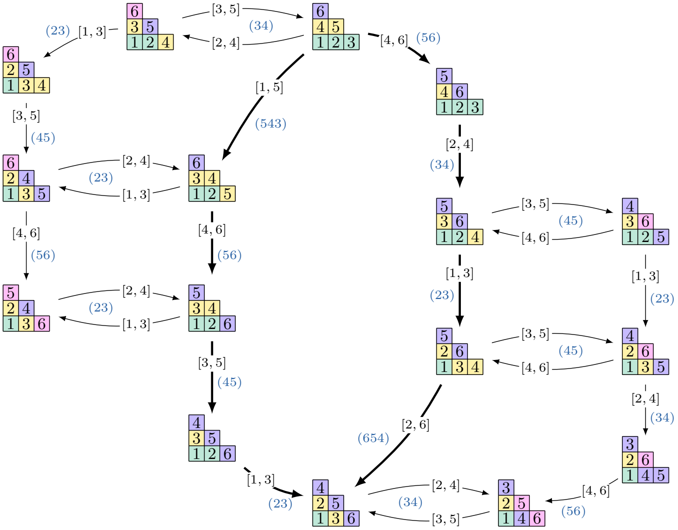
$$P(\pi|_I) = \begin{array}{|c|c|c|c|c} \hline i+m+1 & i+m+2 & \cdots & i+2m & \\ \hline i & i+1 & \cdots & i+m-1 & \multicolumn{1}{c|}{i+m} \\ \hline \end{array}$$

## Example

$\pi = 524136$. The interval $I = [2, 4]$ is a Dyck pattern interval since $\pi|_{[2,4]} = 243$ and

$$P(\pi|_{[2,4]}) = \begin{array}{|c|c} \hline 4 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

# Edges: Dyck pattern intervals



# Edges: Cycles

## Definition

$$\text{cycle}(\pi|_I) := (m + \pi_p, m + \pi_p - 1, \ldots, \pi_p)$$

$\pi_p$ is letter where $f_i$ acts in destandardization and $I = [i, i + 2m]$

# Edges: Cycles

## Definition

$$\text{cycle}(\pi|_I) := (m + \pi_p, m + \pi_p - 1, \ldots, \pi_p)$$

$\pi_p$ is letter where $f_i$ acts in destandardization and $I = [i, i + 2m]$

## Example

$I = [1, 5]$ $\quad T = \begin{array}{|c|c|c|}\hline 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline \end{array}$ $\quad \text{destand}(T) = \begin{array}{|c|c|c|}\hline 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 1 \\ \hline \end{array}$ $\quad \pi = 645123$

# Edges: Cycles

## Definition

$$\text{cycle}(\pi|_I) := (m + \pi_p, m + \pi_p - 1, \ldots, \pi_p)$$

$\pi_p$ is letter where $f_i$ acts in destandardization and $I = [i, i+2m]$

## Example

$I = [1,5]$ $\quad T = \begin{array}{|c|c|c|}\hline 6 & & \\ \hline 4 & 5 & \\ \hline 1 & 2 & 3 \\ \hline \end{array}$ $\quad \text{destand}(T) = \begin{array}{|c|c|c|}\hline 6 & & \\ \hline 2 & 2 & \\ \hline 1 & 1 & 1 \\ \hline \end{array}$ $\quad \pi = 645123$

Crystal operator $f_1$ acts on rightmost 1 and hence $\pi_p = 3$

# Edges: Cycles

## Definition

$$\mathrm{cycle}(\pi|_I) := (m + \pi_p, m + \pi_p - 1, \ldots, \pi_p)$$

$\pi_p$ is letter where $f_i$ acts in destandardization and $I = [i, i + 2m]$

## Example

$I = [1, 5]$ $\quad T = \begin{array}{|c|c|c|}\hline 6 \\ \cline{1-1} 4 & 5 \\ \cline{1-2} 1 & 2 & 3 \\ \hline \end{array}$ $\quad \mathrm{destand}(T) = \begin{array}{|c|c|c|}\hline 6 \\ \cline{1-1} 2 & 2 \\ \cline{1-2} 1 & 1 & 1 \\ \hline \end{array}$ $\quad \pi = 645123$

Crystal operator $f_1$ acts on rightmost 1 and hence $\pi_p = 3$
Dyck pattern interval $I = [1, 5]$

# Edges: Cycles

## Definition

$$\text{cycle}(\pi|_I) := (m + \pi_p, m + \pi_p - 1, \ldots, \pi_p)$$

$\pi_p$ is letter where $f_i$ acts in destandardization and $I = [i, i + 2m]$

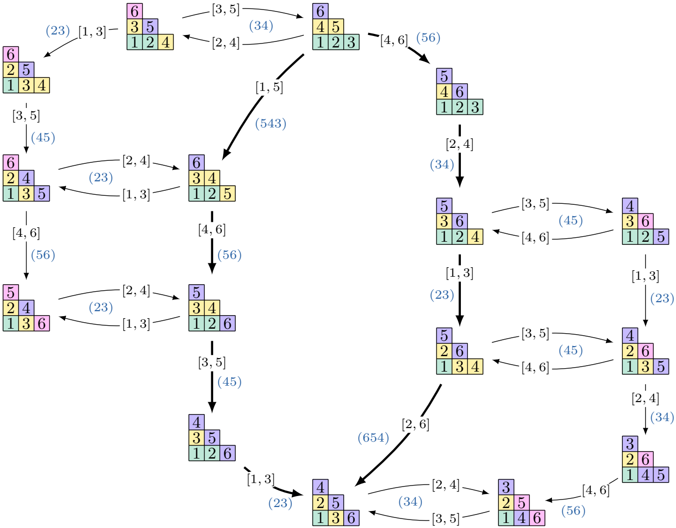
## Example

$I = [1,5]$ $\quad T = \begin{array}{|c|c|c|}\hline 6 \\ \hline 4 & 5 \\ \hline 1 & 2 & 3 \\ \hline\end{array}$ $\quad \text{destand}(T) = \begin{array}{|c|c|c|}\hline 6 \\ \hline 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline\end{array}$ $\quad \pi = 645123$

Crystal operator $f_1$ acts on rightmost 1 and hence $\pi_p = 3$
Dyck pattern interval $I = [1,5]$
Cycle $\text{cycle}(\pi|_I) = (5,4,3)$

$$(5,4,3) \cdot T = T'$$

# Edges: Cycles



# Length of descent composition

**Theorem (BCDS'25)**

*Edge in* $\mathrm{CS}(\lambda)$

$$(T, \alpha) \xrightarrow{I} (T', \beta)$$

$\alpha$ *of length* $\ell \Rightarrow \beta$ *of length* $\ell - 1, \ell$ *or* $\ell + 1$

# Properties: $S_n$-branching

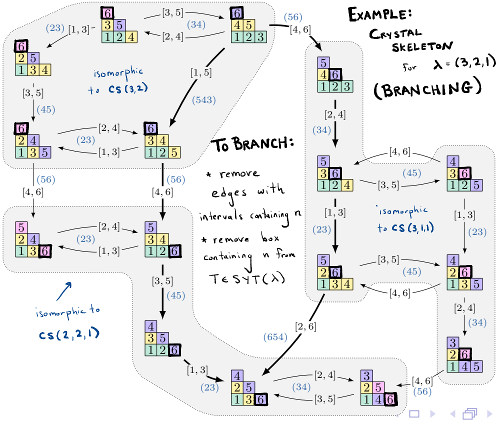
## Theorem (BCDS'25)

$$\mathrm{CS}(\lambda)_{[1,n-1]} \cong \bigoplus_{\lambda^-} \mathrm{CS}(\lambda^-)$$

# Properties: $S_n$-branching

## Theorem (BCDS'25)

$$\mathrm{CS}(\lambda)_{[1,n-1]} \cong \bigoplus_{\lambda^-} \mathrm{CS}(\lambda^-)$$



# Properties: Subcrystal

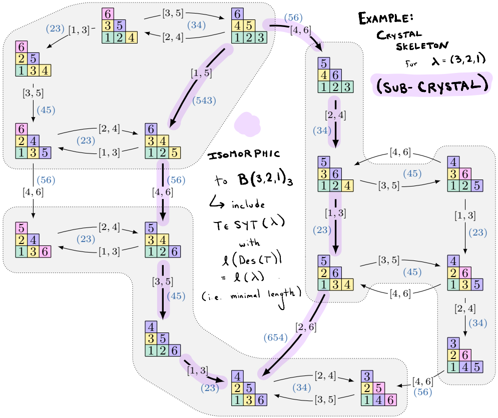
**Theorem (BCDS'25)**

$\mathrm{CS}(\lambda)$ *contains crystal* $B(\lambda)$ *as a subgraph.*

# Properties: Subcrystal

## Theorem (BCDS'25)

$\mathrm{CS}(\lambda)$ *contains crystal* $B(\lambda)$ *as a subgraph.*



# Properties: Dual equivalence subgraph

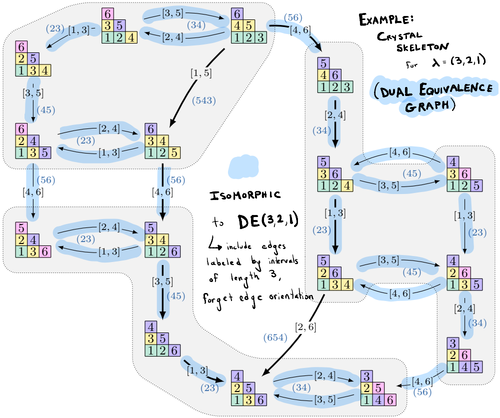
**Theorem (BCDS'25 proving conjecture of Maas-Gariépy)**

*The dual equivalence graph* $\mathrm{DE}(\lambda)$ *is a subgraph of* $\mathrm{CS}(\lambda)$.

# Properties: Dual equivalence subgraph

**Theorem (BCDS'25 proving conjecture of Maas-Gariépy)**

*The dual equivalence graph* DE($\lambda$) *is a subgraph of* CS($\lambda$).



# Properties: Lusztig involution

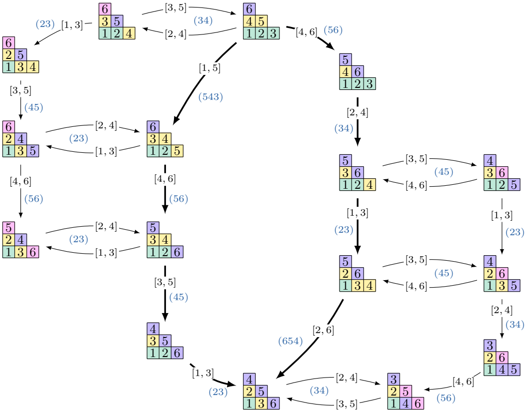
**Theorem (BCDS'25)**

$\mathrm{CS}(\lambda)$ *is symmetric under Lusztig involution:* $\mathrm{CS}(\lambda) \cong \mathcal{L}_n(\mathrm{CS}(\lambda))$.

# Properties: Lusztig involution

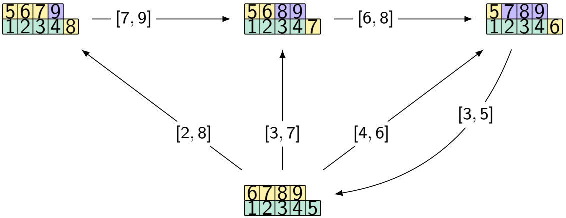
**Theorem (BCDS'25)**

$\mathrm{CS}(\lambda)$ *is symmetric under Lusztig involution:* $\mathrm{CS}(\lambda) \cong \mathcal{L}_n(\mathrm{CS}(\lambda))$.



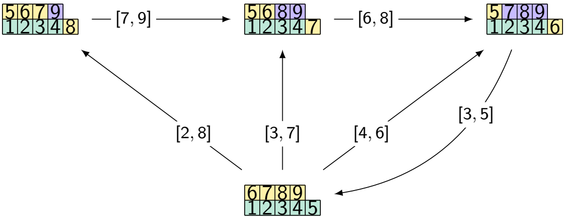
# Properties: Fans

## Fans for length increasing edges:



# Properties: Fans

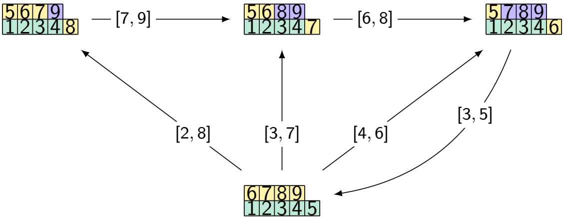
Fans for length increasing edges:




- Nested intervals up

# Properties: Fans

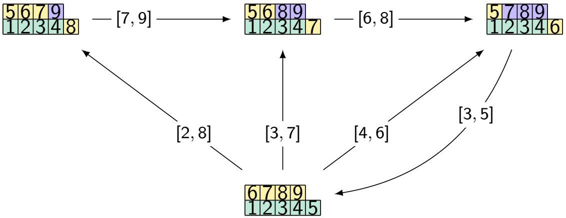
## Fans for length increasing edges:




- Nested intervals up
- Length 3 intervals across

# Properties: Fans

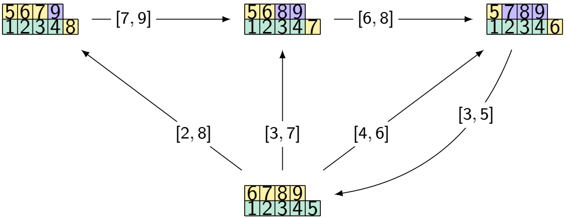
## Fans for length increasing edges:




- Nested intervals up
- Length 3 intervals across
- Length 3 loop on the right

# Properties: Fans

Fans for length increasing edges:




- Nested intervals up
- Length 3 intervals across
- Length 3 loop on the right

Similarly for length decreasing edges.

# Axiomatic description

1. $GL_n$-**axioms**:
   - Strong Lusztig involution: $G \cong \mathcal{L}_n(G)$ and $G_{[1,n-1]} \cong \mathcal{L}_{n-1}(G_{[1,n-1]})$
   - Subcrystal
   - Fans

# Axiomatic description

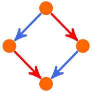
1. $GL_n$-**axioms**:
   - Strong Lusztig involution: $G \cong \mathcal{L}_n(G)$ and $G_{[1,n-1]} \cong \mathcal{L}_{n-1}(G_{[1,n-1]})$
   - Subcrystal
   - Fans
2. $S_n$-**axioms**:
   - Lusztig involution: $G \cong \mathcal{L}_n(G)$
   - $S_n$-branching
   - Connectivity
   - Fans

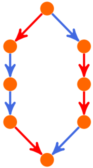
# Axiomatic description

1. $GL_n$-**axioms**:
   - Strong Lusztig involution: $G \cong \mathcal{L}_n(G)$ and $G_{[1,n-1]} \cong \mathcal{L}_{n-1}(G_{[1,n-1]})$
   - Subcrystal
   - Fans
2. $S_n$-**axioms**:
   - Lusztig involution: $G \cong \mathcal{L}_n(G)$
   - $S_n$-branching
   - Connectivity
   - Fans
3. **Local axioms**:
   - Commutation relations (à la Stembridge):
     triangles, squares, pentagons, octagons + fans

# Local characterization of crystals

Local characterization of simply-laced crystals associated to representations (Stembridge 2003)

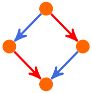


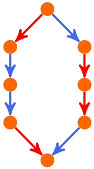


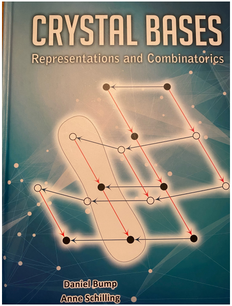
# Local characterization of crystals

Local characterization of simply-laced crystals associated to representations (Stembridge 2003)

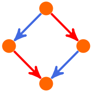
Combinatorial theory of crystals without quantum groups:

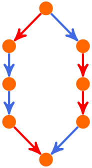




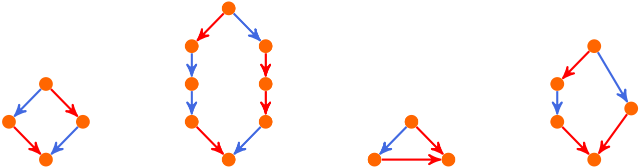


# Local characterization of crystal skeletons





# Local characterization of crystal skeletons



$+$ fans

# Outline

# Motivation

## Tiling and contraction

| | | | | |
|---|---|---|---|---|
| Crystal $B(\lambda)$<br>$\mathrm{char}B(\lambda) = s_\lambda$<br>$\cup\!\mid$<br>Quasicrystal $Q_\beta$<br>$\mathrm{char}Q_\beta = F_\beta$ | $\longrightarrow$ | Crystal skeleton $\mathrm{CS}(\lambda)$<br>$\mathrm{charCS}(\lambda) = s_\lambda$<br>$\cup\!\mid$<br>Quasicrystal skeleton $\mathrm{QCS}_\alpha$<br>$\mathrm{charQCS}_\alpha = \mathcal{YQS}_\alpha$ | $\longrightarrow$ | Bruhat order |

# Young quasisymmetric Schur function

2011 Haglund, Luoto, Mason, van Willigenburg
2013 Luoto, Mykytiuk, van Willigenburg

Young quasisymmetric Schur function

$$\mathcal{YQS}_\alpha = \sum_{T \in \mathrm{SSYCT}(\alpha)} x^T$$

# Young quasisymmetric Schur function

2011 Haglund, Luoto, Mason, van Willigenburg
2013 Luoto, Mykytiuk, van Willigenburg

Young quasisymmetric Schur function

$$\mathcal{YQS}_\alpha = \sum_{T \in \mathrm{SSYCT}(\alpha)} x^T$$

## Definition

Semistandard Young composition tableau (SSYCT) of shape $\alpha$:
Filling $T$ of the cells of $\alpha$ such that

1. Rows are weakly increasing.
2. Leftmost column is strictly increasing.
3. Triple condition: If $a \geqslant b$ then $a > c$.

$$\begin{array}{|c|c|} \hline b & c \\ \hline \end{array}$$
$$\begin{array}{c|c|} \cline{2-2} & a \\ \cline{2-2} \end{array}$$

# Bijection

## Example

Standard Young composition tableau

$$C = \begin{array}{|c|c|c|}\cline{1-2} 4 & 6 \\ \cline{1-2} 3 \\ \hline 1 & 2 & 5 \\ \hline \end{array}$$

# Bijection

## Example

Standard Young composition tableau

$$C = \begin{array}{|c|c|c}\hline 4 & 6 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{|c}{} & \\ \hline 1 & 2 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad T = \begin{array}{|c|c|c}\hline 4 & \multicolumn{1}{|c}{} & \\ \hline 3 & 6 & \multicolumn{1}{|c}{} \\ \hline 1 & 2 & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

## Definition

Bijection

$$\begin{aligned} \varphi\colon \mathsf{SYCT}(\alpha) &\to \mathsf{SYT}(\lambda) \\ C &\mapsto T \end{aligned}$$

# Bijection

## Example

Standard Young composition tableau

$$C = \begin{array}{|c|c|c|}\cline{1-2} 4 & 6 \\ \cline{1-2} 3 \\ \hline 1 & 2 & 5 \\ \hline\end{array} \qquad T = \begin{array}{|c|c|c|}\cline{1-1} 4 \\ \cline{1-2} 3 & 6 \\ \hline 1 & 2 & 5 \\ \hline\end{array}$$

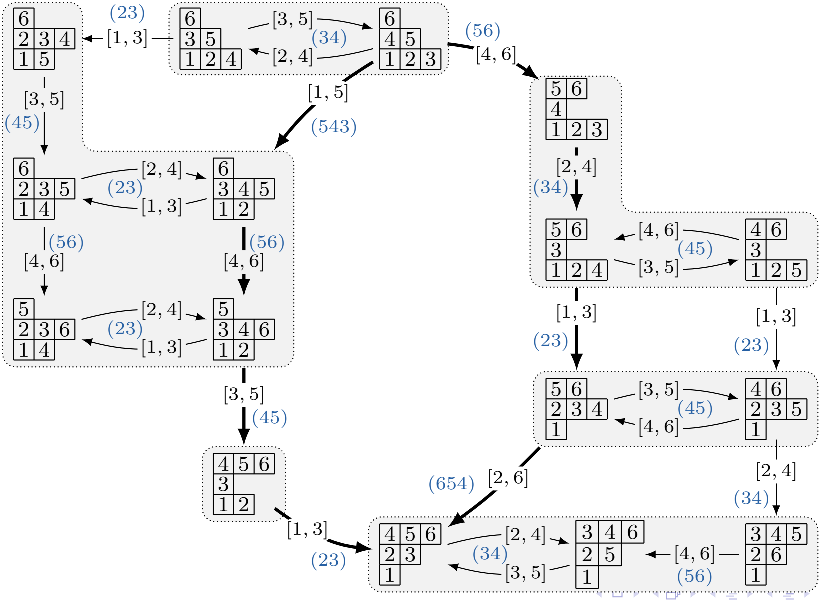
## Definition

### Bijection

$$\varphi\colon \mathsf{SYCT}(\alpha) \to \mathsf{SYT}(\lambda)$$
$$C \mapsto T$$

- $\varphi$ reorder columns of $C$
- $\varphi^{-1}$: first column of $T$, place subsequent letters as high as possible

# Quasicrystal skeleton



# Properties of quasicrystal skeleton

## Definition

Quasicrystal skeleton $\mathrm{QCS}_\alpha$ is induced subgraph of $\mathrm{CS}(\lambda)$ consisting of all vertices $T \in \mathrm{SYT}(\lambda)$ such that $\varphi^{-1}(T)$ has shape $\alpha$.

# Properties of quasicrystal skeleton

## Definition

Quasicrystal skeleton $\mathrm{QCS}_\alpha$ is induced subgraph of $\mathrm{CS}(\lambda)$ consisting of all vertices $T \in \mathrm{SYT}(\lambda)$ such that $\varphi^{-1}(T)$ has shape $\alpha$.

Character

$$\mathrm{char}\mathrm{QCS}_\alpha = \mathcal{YQS}_\alpha$$

# Properties of quasicrystal skeleton

## Definition

Quasicrystal skeleton $\mathrm{QCS}_\alpha$ is induced subgraph of $\mathrm{CS}(\lambda)$ consisting of all vertices $T \in \mathrm{SYT}(\lambda)$ such that $\varphi^{-1}(T)$ has shape $\alpha$.

Character

$$\mathrm{charQCS}_\alpha = \mathcal{YQS}_\alpha$$

## Theorem (BDMS2025)

*Characterization of edges* $T \xrightarrow{I} T'$ *in crystal skeleton* $\mathrm{CS}(\lambda)$ *which change quasicrystal skeleton components:*

- $T \in \mathrm{QCS}_\alpha$ *and* $T' \in \mathrm{QCS}_\beta$ *with* $\alpha \neq \beta$

# Applications

# Applications

- Stanley symmetric functions (Brauner, Daugherty, Mason)

# Applications

- Stanley symmetric functions (Brauner, Daugherty, Mason)
- LLT polynomials (Brauner, Corteel, Daugherty)

# Applications

- Stanley symmetric functions (Brauner, Daugherty, Mason)
- LLT polynomials (Brauner, Corteel, Daugherty)
- Thrall's problem (Pappe, Simone)

# Applications

- Stanley symmetric functions (Brauner, Daugherty, Mason)
- LLT polynomials (Brauner, Corteel, Daugherty)
- Thrall's problem (Pappe, Simone)
- plethysm, Kronecker, Macdonald, ....

# Applications

- Stanley symmetric functions (Brauner, Daugherty, Mason)
- LLT polynomials (Brauner, Corteel, Daugherty)
- Thrall's problem (Pappe, Simone)
- plethysm, Kronecker, Macdonald, ....

Thank you!