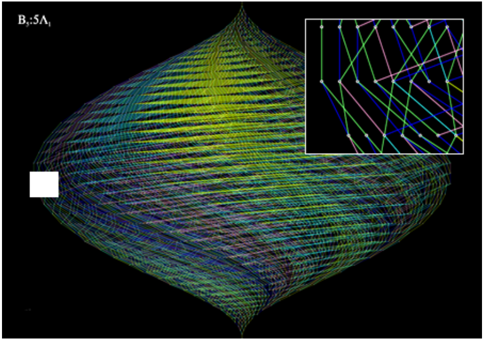
The Ubiquity of Crystal Bases

Anne Schilling

Department of Mathematics, UC Davis



Noether Lecture
Joint Meetings San Francisco
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Crystal bases

Ten reasons why the combinatorial theory of crystal bases which originated in statistical mechanics and quantum groups is ubiquitous in representation theory,combinatorics, geometry, and beyond.

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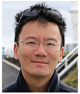
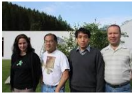
Based on work with my many collaborators over the years:
Assaf, Bandlow, Benkart, Bump, Colmenarejo, Deka, Fourier, Gillespie, Harris, Hawkes, Hersh, Jones, Kirillov, Lam, Lenart, Morse, Naito, Okado, Orellana, Pan, Panova, Pappe, Paramonov, Paul, Pfannerer, Poh, Sagaki, Sakamoto, Saliola, Scrimshaw, Shimozono, Simone, Sternberg, Thiéry, Tingley, Wang, Warnaar, Yip, Zabrocki

Outline

Lie algebras

Lie algebra $\mathfrak{sl}_2$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Relations

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = h$$

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$$eV(\lambda) \subset V(\lambda + 2) \qquad fV(\lambda) \subset V(\lambda - 2)$$

Quantum groups

Quantum group $U_q(\mathfrak{sl}_2)$

generated by $e, f, K^{\pm 1}$

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Representations

$(m+1)$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-representation

$$V_{(m)} = \{u, f^{(1)}u, \ldots, f^{(m)}u\}$$

$$\text{where} \quad eu = 0 \quad Ku = q^m u \quad f^{(k)}u = \frac{1}{[k]_q!} f^k u \quad [k]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$$

Motivation for crystal bases

2-dimensional $U_q(\mathfrak{sl}_2)$-representation $V_{(1)}$

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$$\mathrm{u} \underset{e}{\overset{f}{\rightleftarrows}} \mathrm{v}$$

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Basis for $V_{(1)} \otimes V_{(1)}$ is $u \otimes u, v \otimes u, u \otimes v, v \otimes v$

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$$V_{(1)} \otimes V_{(1)} \cong V_{(2)} \oplus V_{(0)} \qquad V_{(2)} = \{u \otimes u, u \otimes v + qv \otimes u, v \otimes v\}$$

$$V_{(0)} = \{v \otimes u - qu \otimes v\}$$

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Crystal basis

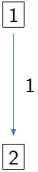
Pick leading term ($q \to 0$)

$$B_{(1)} \otimes B_{(1)} \cong B_{(2)} \oplus B_{(0)} \qquad B_{(2)} = \{u \otimes u, u \otimes v, v \otimes v\}$$
$$B_{(0)} = \{v \otimes u\}$$

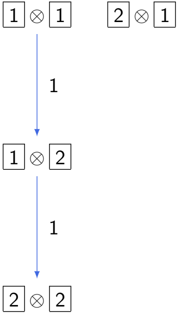
Motivation for crystal bases

$u = \boxed{1}$ $\quad v = \boxed{2}$

$B_{\square}$



$B_{\square} \otimes B_{\square}$



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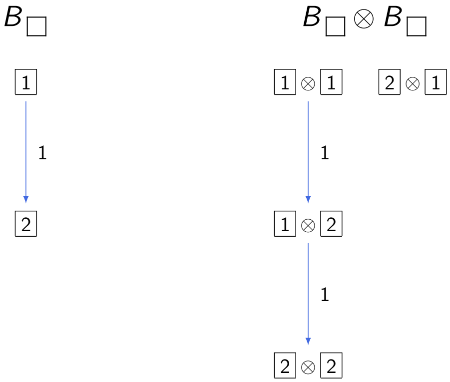
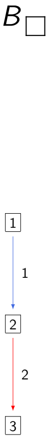


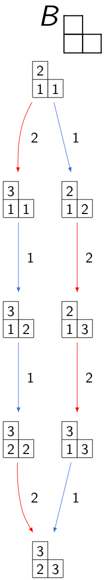
Reason 1

Crystal bases are combinatorial skeletons of representation theory.

Outline

$U_q(\mathfrak{sl}_3)$-crystals





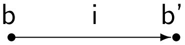
Axiomatic Crystals

A $U_q(\mathfrak{g})$-crystal is a nonempty set B with maps

$$\mathrm{wt} : B \to P$$
$$e_i, f_i : B \to B \cup \{\emptyset\} \quad \text{for all } i \in I$$

satisfying

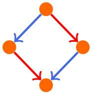
$$f_i(b) = b' \Leftrightarrow e_i(b') = b \qquad \text{if } b, b' \in B$$
$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \qquad \text{if } f_i(b) \in B$$
$$\langle h_i, \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b)$$

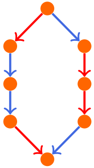
Write 

for $b' = f_i(b)$

Local characterization

Local characterization of simply-laced crystals associated to representations (Stembridge 2003)





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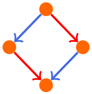
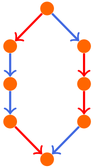
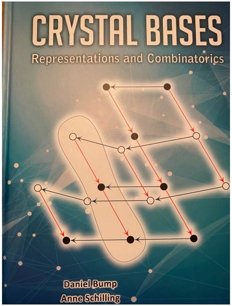
Combinatorial theory of crystals without quantum groups:







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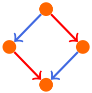

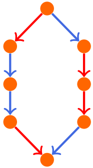

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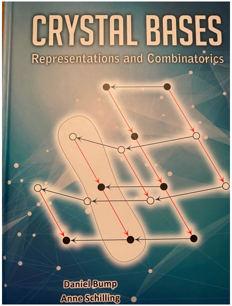
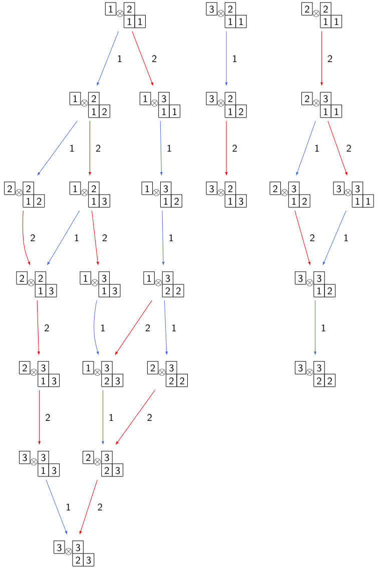


Reason 2

Crystal graphs can be characterized by local combinatorial rules.

Tensor product decomposition



$B_{\square} \otimes B_{\tiny\begin{array}{|c|c}\hline \square & \\ \hline \square & \square \\ \hline\end{array}}$

Tensor products of crystals

Definition

B, B' crystals

$B \otimes B'$ is $B \times B'$ as sets with

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$$

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geqslant \varphi_i(b') \\ b \otimes f_i(b') & \text{otherwise} \end{cases}$$

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Reason 3

Crystals are well behaved with respect to tensor products.

Tensor product multiplicities

- Irreducible $\mathfrak{sl}_n$-representation

$$V_\lambda$$

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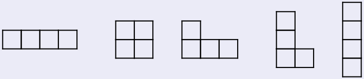
$$V_\lambda$$

Indexed by partitions: 

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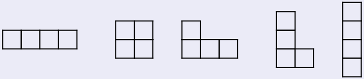
Indexed by partitions: 

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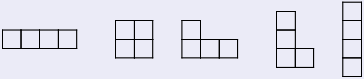
Indexed by partitions: 

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Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$

Combinatorial description

Littlewood–Richardson rule

$c^{\nu}_{\lambda\mu} = \#$ skew tableaux t of shape ν/λ and weight μ such that $\mathrm{row}(t)$ is a reverse lattice word.

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Example

$$V_{(2,1)} \otimes V_{(2,1)} = \cdots \oplus ?V_{(3,2,1)} \oplus \cdots$$

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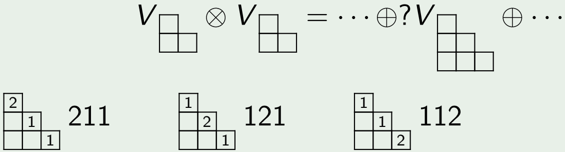

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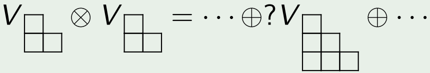

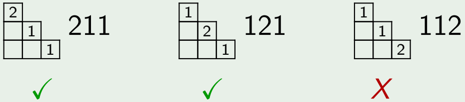

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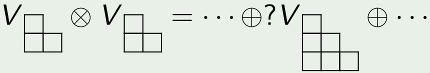

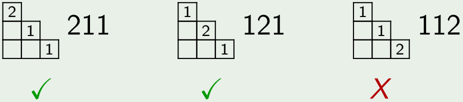

$\Rightarrow c^{321}_{21,21} = 2$

Gordon James (1987) on the Littlewood-Richardson rule:

> *"Unfortunately the Littlewood-Richardson rule is much harder to prove than was at first suspected. The author was once told that the Littlewood-Richardson rule helped to get men on the moon but was not proved until after they got there."*

Crystal graph

Action of crystal operators e_i, f_i on tableaux:

1. Consider letters i and $i+1$ in row reading word of the tableau
2. Successively "bracket" pairs of the form $(i+1, i)$
3. Left with word of the form $i^r(i+1)^s$

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$$e_i(i^r(i+1)^s) = \begin{cases} i^{r+1}(i+1)^{s-1} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

$$f_i(i^r(i+1)^s) = \begin{cases} i^{r-1}(i+1)^{s+1} & \text{if } r > 0 \\ 0 & \text{else} \end{cases}$$

Crystal reformulation

3							
	1	2	2	3			
		1	1	2	3	3	3

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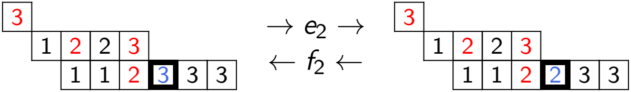
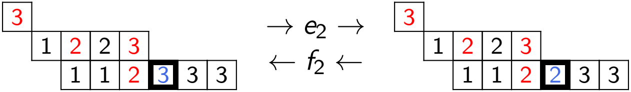
Crystal reformulation



e_2: change leftmost unpaired 3 into 2

f_2: change rightmost unpaired 2 into 3

Crystal reformulation



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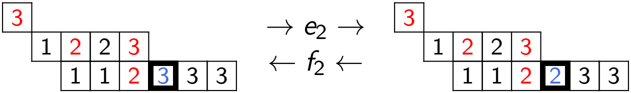
Theorem

b where all $e_i(b) = 0$ (highest weight)

↔ connected component

↔ irreducible

Crystal reformulation



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Theorem

b where all $e_i(b) = 0$ (highest weight)

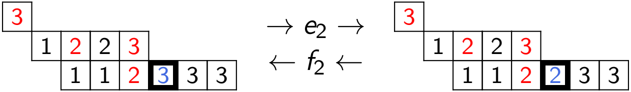
$\leftrightarrow$ *connected component*

$\leftrightarrow$ *irreducible*

Reformulation of LR rule

$c^{\nu}_{\lambda\mu}$ counts tableaux of shape ν/λ and weight μ which are highest weight.

Crystal reformulation



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$\leftrightarrow$ *connected component*
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Reason 4

Crystal operators explain/match the Littlewood–Richardson rule.

Outline

Schur functions

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Definition

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Example

Semi-standard Young tableaux of shape $(2, 1)$ over the alphabet $\{1, 2, 3\}$

$$\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\quad
\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\quad
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Schur functions

B_λ = set of semi-standard Young tableaux of partition shape λ over alphabet $\{1, 2, \ldots, n\}$

Definition

Schur polynomial

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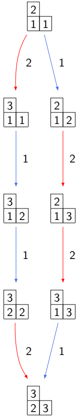
Semi-standard Young tableaux of shape $(2, 1)$ over the alphabet $\{1, 2, 3\}$

$$\begin{array}{cc} 2 & \\ 1 & 1 \end{array} \quad \begin{array}{cc} 3 & \\ 1 & 1 \end{array} \quad \begin{array}{cc} 3 & \\ 2 & 2 \end{array} \quad \begin{array}{cc} 3 & \\ 1 & 2 \end{array} \quad \begin{array}{cc} 2 & \\ 1 & 3 \end{array} \quad \begin{array}{cc} 2 & \\ 1 & 2 \end{array} \quad \begin{array}{cc} 3 & \\ 1 & 3 \end{array} \quad \begin{array}{cc} 3 & \\ 2 & 3 \end{array}$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + 2x_1 x_2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$$

Crystal structure

Crystal $B_{\tiny\yng(1,2)}$ with edges f_1 $\color{blue}{\downarrow}$ and f_2 $\color{red}{\downarrow}$



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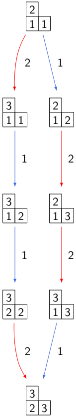
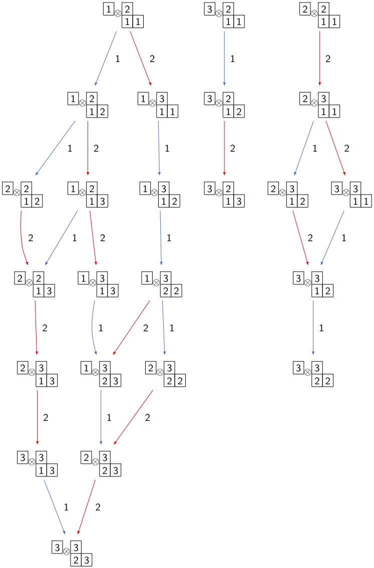


Reason 5

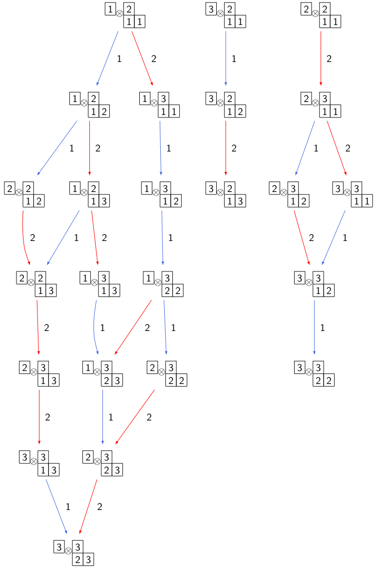
Schur polynomials are characters of type A crystals.

Tensor product decomposition



$$B_{\square} \otimes B_{\substack{\square\\ \square\square}}$$

Tensor product decomposition



$$B_{\square} \otimes B_{(2,1)}$$

$$= B_{(3,1)} \oplus B_{(2,1,1)} \oplus B_{(2,2)}$$

Symmetric functions

Reformulation of LR rule

$c_{\lambda\mu}^{\nu}$ counts pairs of tableaux of shape λ and μ of weight ν which are highest weight.

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Symmetric function coefficients

$$s_{\nu/\mu} = \sum_{\lambda} c^{\nu}_{\lambda\mu} s_{\lambda} \quad \text{and} \quad s_{\lambda} s_{\mu} = \sum_{\nu} c^{\nu}_{\lambda\mu} s_{\nu}$$

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Mechanism to get Schur expansion

$$s_{\nu/\lambda} = \sum_{T\in B_{\nu/\lambda}} x^{weight(T)} = \sum_{YT=highest\ \ weights} s_{weight(YT)}$$

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Reason 6

Crystals can help to understand symmetric functions.

Super Lie algebras

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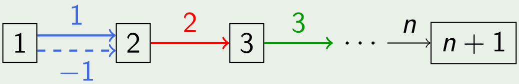
  - Highest weight crystals for queer super Lie algebras (Grantcharov, Jung, Kang, Kashiwara, Kim, '10)

Standard crystal and tensor product

Example

Standard queer crystal $\mathcal{B}$ for $\mathfrak{q}(n+1)$



Standard crystal and tensor product

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$$\boxed{1} \overset{1}{\underset{-1}{\rightrightarrows}} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n} \boxed{n+1}$$

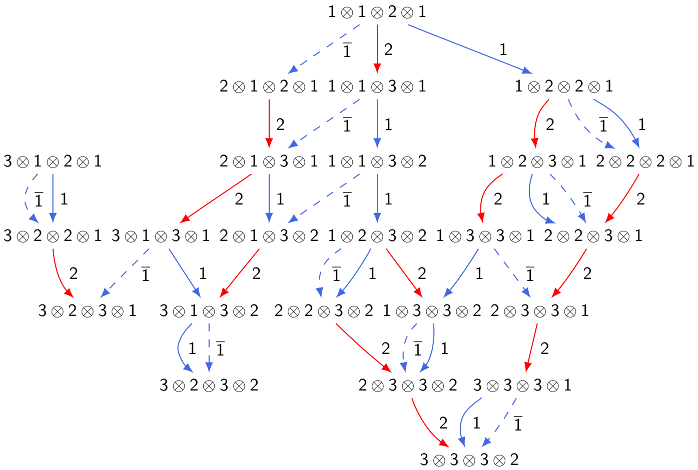
Tensor product: $b \otimes c \in B \otimes C$

$$f_{-1}(b \otimes c) = \begin{cases} b \otimes f_{-1}(c) & \text{if } \mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0 \\ f_{-1}(b) \otimes c & \text{otherwise} \end{cases}$$

$$e_{-1}(b \otimes c) = \begin{cases} b \otimes e_{-1}(c) & \text{if } \mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0 \\ e_{-1}(b) \otimes c & \text{otherwise} \end{cases}$$

Queer crystal: Example

One connected component of $\mathcal{B}^{\otimes 4}$ for $\mathfrak{q}(3)$:



Motivation

Why are queer crystals interesting?

- Characters:
  character of highest weight crystal B_λ (λ strict partition) is Schur P function P_λ

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- Littlewood–Richardson rule:

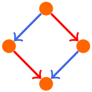
$$P_\lambda P_\mu = \sum_\nu g_{\lambda\mu}^\nu P_\nu$$

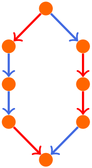
$g_{\lambda\mu}^\nu$ = number of highest weights of weight ν in $B_\lambda \otimes B_\mu$

Characterizations

Characterization of crystals:

- Local characterization of simply-laced crystals (Stembridge 2003)





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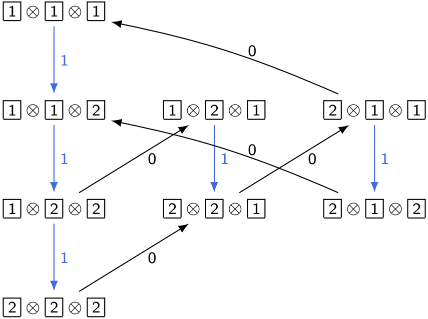
Reason 7

Crystals provide combinatorial analysis of super Lie algebras.

Outline

Affine crystals



One dimensional configuration sums

Why affine crystals?

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- energy function $E : B_N \otimes \cdots \otimes B_1 \to \mathbb{Z}$

$$E(e_i(b)) = E(b) \qquad \text{for } 1 \leqslant i \leqslant n$$
$$E(e_0(b)) = E(b) - 1$$

if e_0 does not act on leftmost step in $b = b_N \otimes \cdots \otimes b_1$.

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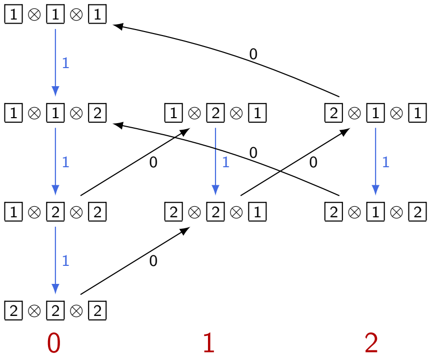
- characters of conformal field theories as limits of $X(\lambda, B)$

Energy function



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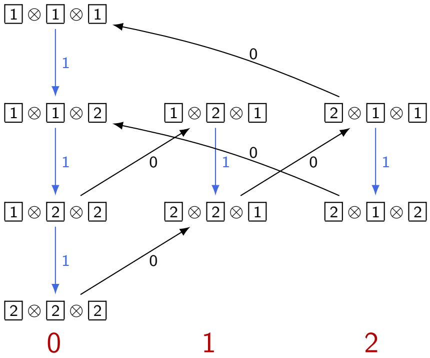
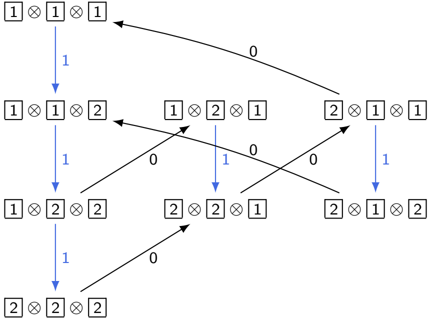


$$X((2,1),B) = 1 + q + q^2$$

Energy function



0 1 2

$X((2,1),B) = 1 + q + q^2$

Reason 8

Affine crystals give the energy function and one-dimensional configuration sums.

Promotion

Crystal commutor: (Henriquez, Kamnitzer 2006)

$$\begin{aligned} \sigma_{B,C} : B \otimes C &\to C \otimes B \\ b \otimes c &\mapsto \eta(\eta(c) \otimes \eta(b)) \end{aligned}$$

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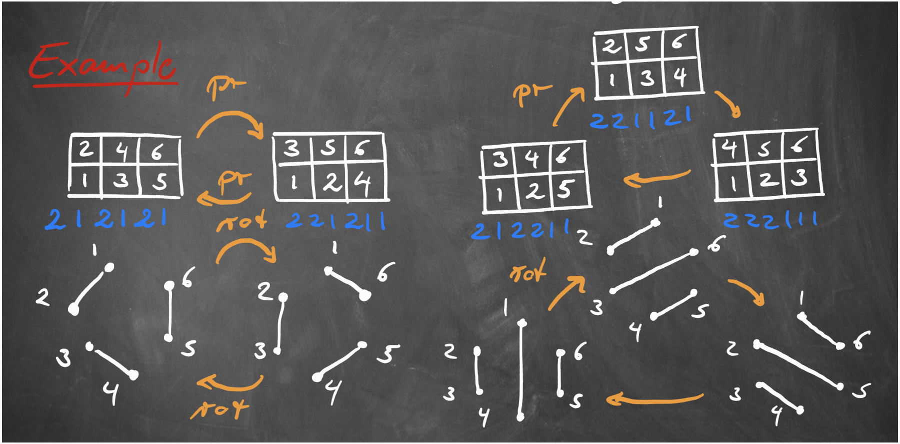
Definition (Promotion)

$u \in B^{\otimes n}$ highest weight

$$\mathrm{pr}(u) = \sigma_{C^{\otimes n-1},C}(u)$$

cyclic action on highest weight elements

Promotion – example



Cyclic sieving phenomenon

Theorem (Fontaine, Kamnitzer 2016, Westbury 2016, Pappe, Pfannerer, S., Simone 2023)

Highest weight elements in $B^{\otimes n}$ of weight zero, promotion, one-dimensional configuration sums gives rise to cyclic sieving phenomenon.

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Cyclic sieving phenomenon: polynomials evaluated at roots of unity related to sizes of orbits under cyclic action

Reason 9

Crystals gives rise to cyclic sieving phenomena and promotion gives a cyclic action.

Thank you !

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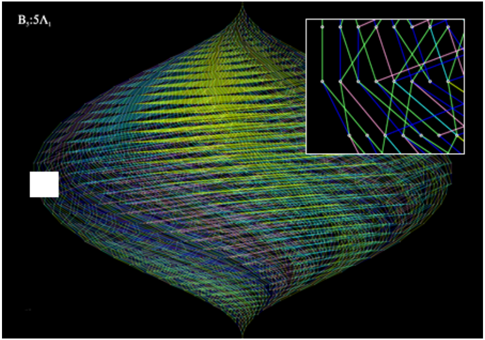

Reason 10

Crystals are beautiful!