The Ubiquity of Crystal Bases

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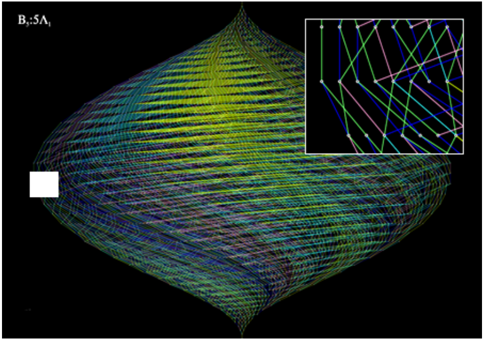

Integrable combinatorics
IMJ-PRG summer school 2026
Paris
June 17, 2026

Crystal bases

Ten reasons why the combinatorial theory of crystal bases
which originated in statistical mechanics and quantum groups
is ubiquitous in representation theory,combinatorics, geometry, and
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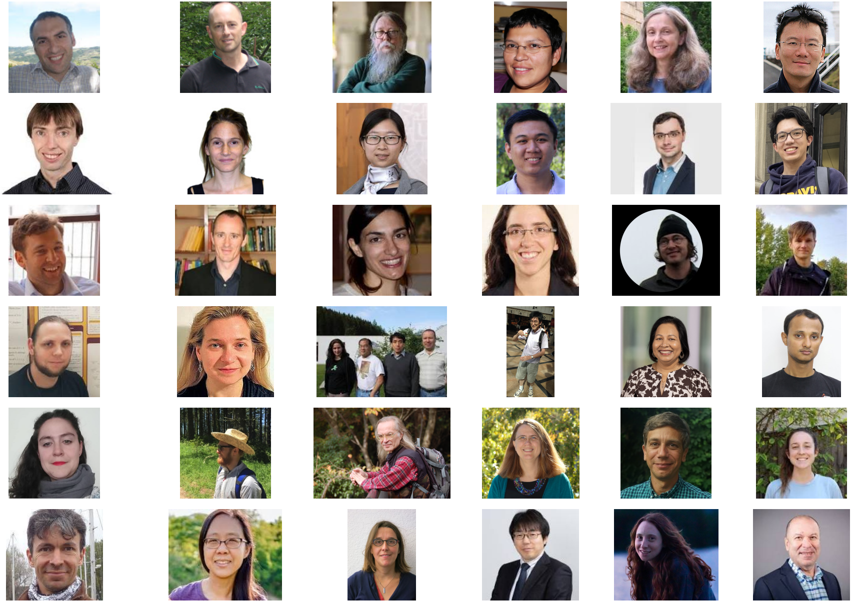
Many crystal collaborators over the years:
Assaf, Bandlow, Benkart, Brauner, Bump, Colmenarejo, Corteel, Daugherty, Deka, Fourier, Gillespie, Harris, Hawkes, Hersh, Jones, Kirillov, Lam, Lenart, Mason, Morse, Naito, Okado, Orellana, Pan, Panova, Pappe, Paramonov, Paul, Pfannerer, Poh, Sagaki, Sakamoto, Saliola, Scrimshaw, Shimozono, Simone, Sternberg, Thiéry, Tingley, Wang, Warnaar, Yip, Zabrocki



Outline

Lie algebras

Lie algebra $\mathfrak{sl}_2$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Relations

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = h$$

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$$eV(\lambda) \subset V(\lambda + 2) \qquad fV(\lambda) \subset V(\lambda - 2)$$

Quantum groups

Quantum group $U_q(\mathfrak{sl}_2)$

generated by $e, f, K^{\pm 1}$

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Representations

$(m+1)$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-representation

$$V_{(m)} = \{u, f^{(1)}u, \ldots, f^{(m)}u\}$$

where $\quad eu = 0 \quad Ku = q^m u \quad f^{(k)}u = \dfrac{1}{[k]_q!} f^k u \quad [k]_q = \dfrac{q^m - q^{-m}}{q - q^{-1}}$

Motivation for crystal bases

2-dimensional $U_q(\mathfrak{sl}_2)$-representation $V_{(1)}$

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$$\begin{tikzcd} u \arrow[r, shift left, "f"] & v \arrow[l, shift left, "e"] \end{tikzcd}$$

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Basis for $V_{(1)} \otimes V_{(1)}$ is $u \otimes u, v \otimes u, u \otimes v, v \otimes v$

$$V_{(1)} \otimes V_{(1)} \cong V_{(2)} \oplus V_{(0)} \qquad V_{(2)} = \{u \otimes u, u \otimes v + qv \otimes u, v \otimes v\}$$

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Crystal basis

Pick leading term ($q \to 0$)

$$B_{(1)} \otimes B_{(1)} \cong B_{(2)} \oplus B_{(0)} \qquad \begin{aligned} B_{(2)} &= \{u \otimes u, u \otimes v, v \otimes v\} \\ B_{(0)} &= \{v \otimes u\} \end{aligned}$$

Motivation for crystal bases

$u = \boxed{1}$ $\quad v = \boxed{2}$

$$B_{\square}$$

$$\begin{tikzcd} \boxed{1} \arrow[d, "1"] \\ \boxed{2} \end{tikzcd}$$

$$B_{\square} \otimes B_{\square}$$

$$\begin{tikzcd} \boxed{1} \otimes \boxed{1} \arrow[d, "1"] & \boxed{2} \otimes \boxed{1} \\ \boxed{1} \otimes \boxed{2} \arrow[d, "1"] & \\ \boxed{2} \otimes \boxed{2} & \end{tikzcd}$$

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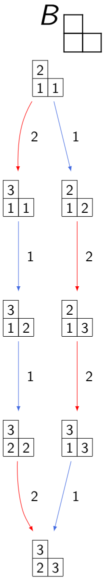
Reason 1

Crystal bases are combinatorial skeletons of representation theory.

Outline

$U_q(\mathfrak{sl}_3)$-crystals



Axiomatic Crystals

A $U_q(\mathfrak{g})$-crystal is a nonempty set B with maps

$$\begin{aligned} \mathrm{wt} &: B \to P \\ e_i, f_i &: B \to B \cup \{\emptyset\} \quad \text{for all } i \in I \end{aligned}$$

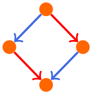
satisfying

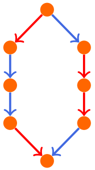
$$\begin{aligned} & f_i(b) = b' \Leftrightarrow e_i(b') = b && \text{if } b, b' \in B \\ & \mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i && \text{if } f_i(b) \in B \\ & \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b) && \end{aligned}$$

Write $\text{b} \xrightarrow{\text{i}} \text{b'}$ for $b' = f_i(b)$

Local characterization

Local characterization of simply-laced crystals associated to representations (Stembridge 2003)





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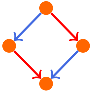
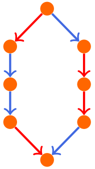
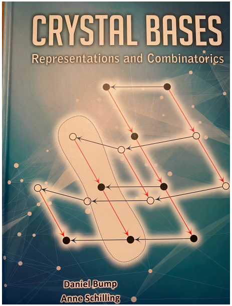




Combinatorial theory of crystals without quantum groups:



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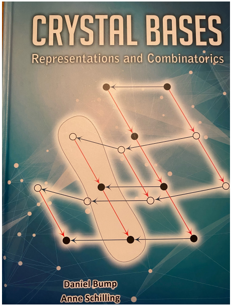
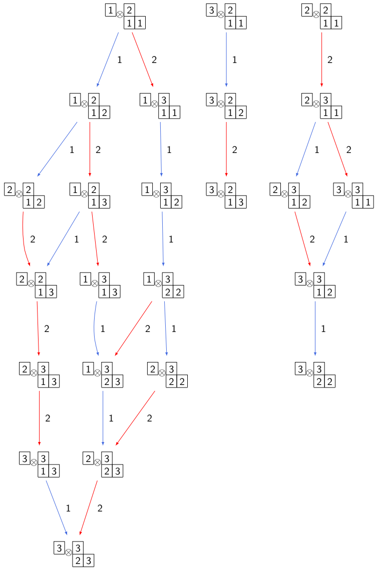


Reason 2

Crystal graphs can be characterized by local combinatorial rules.

Tensor product decomposition



$$B_{\square} \otimes B_{\square\!\!\square\atop\square}$$

Tensor products of crystals

Definition

B, B' crystals

$B \otimes B'$ is $B \times B'$ as sets with

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$$

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geqslant \varphi_i(b') \\ b \otimes f_i(b') & \text{otherwise} \end{cases}$$

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$$\underset{\underbrace{---}_{\varphi_i(b)}\underbrace{+++}_{\varepsilon_i(b)}}{b} \quad \otimes \quad \underset{\underbrace{--}_{\varphi_i(b')}\quad\underbrace{++++}_{\varepsilon_i(b')}}{b'}$$

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$$b \otimes b'$$

$$\underbrace{- - -}_{\varphi_i(b)}\underbrace{+ + {\color{red}+}}_{\varepsilon_i(b)} \qquad \underbrace{{\color{red}-}-}_{\varphi_i(b')} \qquad \underbrace{+ + + +}_{\varepsilon_i(b')}$$

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$$\begin{array}{ccc} b & \otimes & b' \\ \underbrace{- - -}_{\varphi_i(b)} & & \underbrace{--}_{\varphi_i(b')}\ \underbrace{+ + ++}_{\varepsilon_i(b')} \end{array}$$

Reason 3

Crystals are well behaved with respect to tensor products.

Tensor product multiplicities

- Irreducible $\mathfrak{sl}_n$-representation

$$V_\lambda$$

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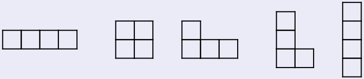
Indexed by partitions: 

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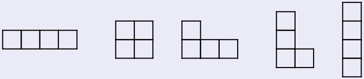
Indexed by partitions: 

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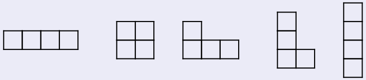
Indexed by partitions: 

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$$V_\lambda \otimes V_\mu = \bigoplus_\nu c_{\lambda\mu}^{\nu} \, V_\nu$$

Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$

Combinatorial description

Littlewood–Richardson rule

$c^{\nu}_{\lambda\mu} = \#$ skew tableaux t of shape ν/λ and weight μ such that $\mathrm{row}(t)$ is a reverse lattice word.

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Example

$$V_{\ydiagram{1,2}} \otimes V_{\ydiagram{1,2}} = \cdots \oplus ?V_{\ydiagram{1,2,3}} \oplus \cdots$$

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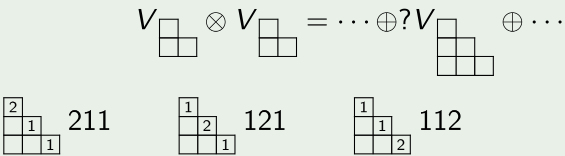

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$$\begin{ytableau} 2 \\ {} & 1 \\ {} & {} & 1 \end{ytableau}\ 211 \qquad \begin{ytableau} 1 \\ {} & 2 \\ {} & {} & 1 \end{ytableau}\ 121 \qquad \begin{ytableau} 1 \\ {} & 1 \\ {} & {} & 2 \end{ytableau}\ 112 \qquad \Rightarrow c^{321}_{21,21} = 2$$

$\checkmark$ $\qquad$ $\checkmark$ $\qquad$ X

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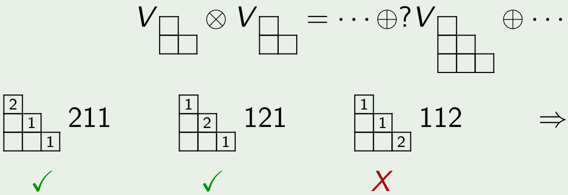

$\Rightarrow c^{321}_{21,21} = 2$

Gordon James (1987) on the Littlewood-Richardson rule:

> *"Unfortunately the Littlewood-Richardson rule is much harder to prove than was at first suspected. The author was once told that the Littlewood-Richardson rule helped to get men on the moon but was not proved until after they got there."*

Crystal graph

Action of crystal operators e_i, f_i on tableaux:

1. Consider letters i and $i+1$ in row reading word of the tableau
2. Successively "bracket" pairs of the form $(i+1, i)$
3. Left with word of the form $i^r(i+1)^s$

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$$e_i(i^r(i+1)^s) = \begin{cases} i^{r+1}(i+1)^{s-1} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

$$f_i(i^r(i+1)^s) = \begin{cases} i^{r-1}(i+1)^{s+1} & \text{if } r > 0 \\ 0 & \text{else} \end{cases}$$

Crystal reformulation

3							
	1	2	2	3			
		1	1	2	3	3	3

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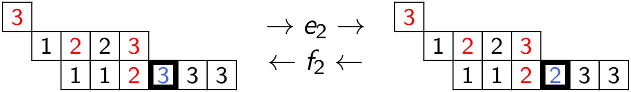
Crystal reformulation



e_2: change leftmost unpaired 3 into 2

f_2: change rightmost unpaired 2 into 3

Crystal reformulation

$$\begin{array}{cccccccc} 3 & & & & & & & \\ & 1 & 2 & 2 & 3 & & & \\ & & 1 & 1 & 2 & \boxed{3} & 3 & 3 \end{array} \quad \begin{array}{c} \rightarrow e_2 \rightarrow \\ \leftarrow f_2 \leftarrow \end{array} \quad \begin{array}{cccccccc} 3 & & & & & & & \\ & 1 & 2 & 2 & 3 & & & \\ & & 1 & 1 & 2 & \boxed{2} & 3 & 3 \end{array}$$

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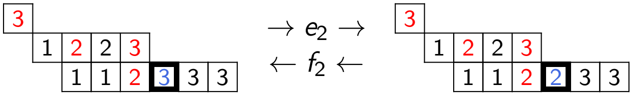
Theorem

b where all $e_i(b) = 0$ (highest weight)

$\leftrightarrow$ *connected component*

$\leftrightarrow$ *irreducible*

Crystal reformulation



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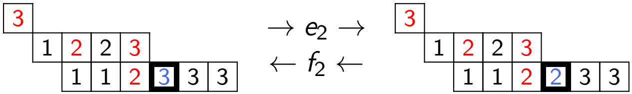
Reformulation of LR rule

$c^{\nu}_{\lambda\mu}$ counts tableaux of shape ν/λ and weight μ which are highest weight.

Crystal reformulation



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Reason 4

Crystal operators explain/match the Littlewood–Richardson rule.

Outline

Schur functions

B_λ = set of semi-standard Young tableaux of partition shape λ
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Definition

Schur polynomial

$$s_\lambda(x) = s_\lambda(x_1, \ldots, x_n) = \sum_{T \in B_\lambda} x_1^{\mathrm{wt}(T)_1} \ldots x_n^{\mathrm{wt}(T)_n}$$

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Example

Semi-standard Young tableaux of shape $(2, 1)$ over the alphabet $\{1, 2, 3\}$

$$\begin{array}{cc} 2 & \\ 1 & 1 \end{array} \quad \begin{array}{cc} 3 & \\ 1 & 1 \end{array} \quad \begin{array}{cc} 3 & \\ 2 & 2 \end{array} \quad \begin{array}{cc} 3 & \\ 1 & 2 \end{array} \quad \begin{array}{cc} 2 & \\ 1 & 3 \end{array} \quad \begin{array}{cc} 2 & \\ 1 & 2 \end{array} \quad \begin{array}{cc} 3 & \\ 1 & 3 \end{array} \quad \begin{array}{cc} 3 & \\ 2 & 3 \end{array}$$

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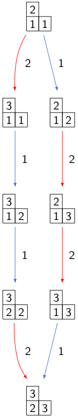
$$\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 3 \\ \hline \end{array}$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + 2x_1 x_2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$$

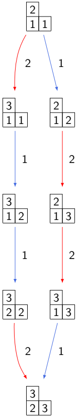
Crystal structure

Crystal $B_{\substack{\square\\\square\square}}$ with edges $f_1 \downarrow$ and $f_2 \downarrow$



Crystal structure

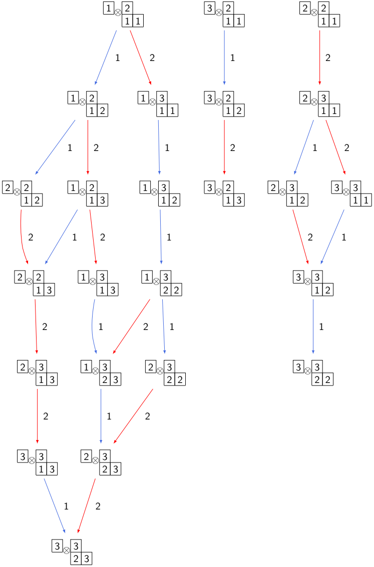
Crystal $B_{\ydiagram{1,2}}$ with edges $f_1 \downarrow$ and $f_2 \downarrow$



Reason 5

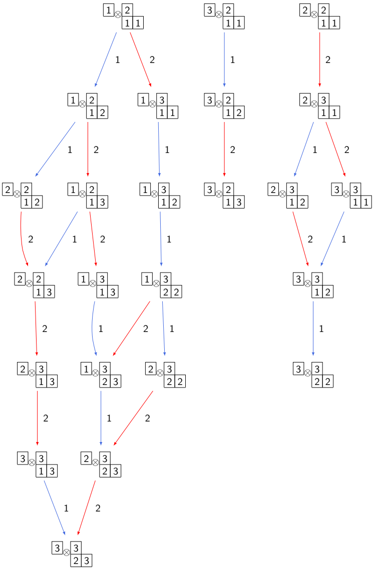
Schur polynomials are characters of type A crystals.

Tensor product decomposition



$$B_{\square} \otimes B_{\substack{\square\\ \square\square}}$$

Tensor product decomposition



$$B_{\ydiagram{1}} \otimes B_{\ydiagram{2,1}}$$

$$= B_{\ydiagram{3,1}} \oplus B_{\ydiagram{2,1,1}} \oplus B_{\ydiagram{2,2}}$$

Symmetric functions

Reformulation of LR rule

$c_{\lambda\mu}^{\nu}$ counts pairs of tableaux of shape λ and μ of weight ν which are highest weight.

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Symmetric function coefficients

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Mechanism to get Schur expansion

$$s_{\nu/\lambda} = \sum_{T \in B_{\nu/\lambda}} x^{weight(T)} = \sum_{YT=highest\ \ weights} s_{weight(YT)}$$

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Reason 6

Crystals can help to understand symmetric functions.

Application: Stanley symmetric functions

History:

- **1984:** Stanley introduced Stanley symmetric functions $\mathcal{F}_w$ to enumerate reduced expressions of permutations

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- **2016:** Morse–S imposed crystal structure on increasing factorization

Reduced factorizations

Definition

- ρ reduced word
  $(S_n = \langle s_1, \ldots, s_{n-1} \mid s_i s_j = s_j s_i, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i^2 = id \rangle)$
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Example

Reduced factorizations for $w = 321$ with reduced words 121, 212:

$$\begin{array}{lllll} ()(12)(1) & (12)()(1) & (12)(1)() & (1)(2)(1) & \text{for } \ell = 3 \\ ()(2)(12) & (2)()(12) & (2)(12)() & (2)(1)(2) & \end{array}$$

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weight $\mathrm{wt}(r)$ of increasing factorization r is weak composition with ith part the number of letters in ith block of r from the right

Example

$\mathrm{wt}(()(12)(1)) = (0, 2, 1)$

Stanley symmetric functions

Definition

Stanley symmetric polynomial indexed by permutation w

$$\mathcal{F}_w(x_1, \ldots, x_\ell) = \sum_r x^{\mathrm{wt}(r)}$$

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$$\begin{aligned}\mathcal{F}_{321}(x_1,x_2,x_3) &= x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + 2x_1x_2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 \\ &= s_{2,1}(x_1,x_2,x_3)\end{aligned}$$

Crystal on increasing factorizations

$$\begin{tikzcd}
& (12)(1)() \arrow[dl, blue] \arrow[dr, red] & \\
(2)(12)() \arrow[d, red] & & (12)()(1) \arrow[d, blue] \\
(2)(1)(2) \arrow[d, red] & & (1)(2)(1) \arrow[d, blue] \\
(2)()(12) \arrow[dr, blue] & & ()(12)(1) \arrow[dl, red] \\
& ()(2)(12) &
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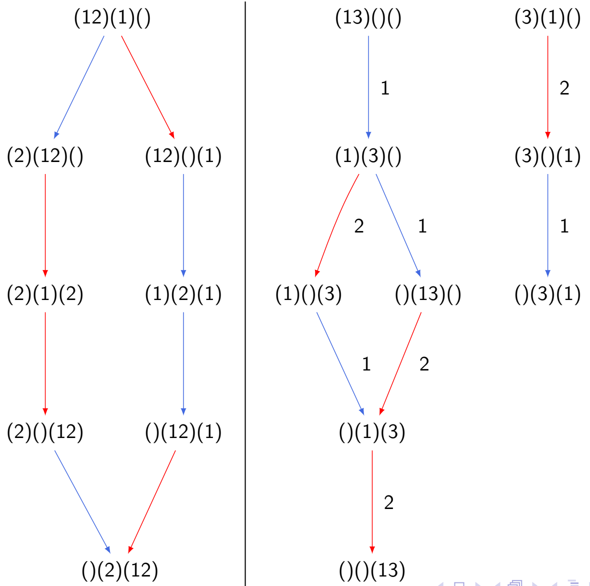
Crystal on increasing factorizations



Stanley crystal

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Crystals can help to understand symmetric functions.

Outline

Variation c_{uv}^{w}

Indexed by permutations: (1,2,3) (2,1,3) (3,2,1) $\cdots$

- Intersections in the set $\mathbb{F}_n$ of complete flags
  $0 = W_0 \subset W_1 \subset \cdots \subset W_n = \mathbb{C}^n$

$$c_{uv}^{w} = X_u \cap X_v \cap X_{w_0 w}$$

- Schubert polynomial coefficients

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WHAT ARE THESE COUNTING?

Schubert polynomials – history

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$$\mathfrak{S}_w(x) = \partial_{w^{-1}w_0}(x_1^{n-1}x_2^{n-2}\cdots x_{n-1})$$

$w \in S_n$, $w_0 = nn-1\ldots 21$ long permutation in S_n

$$\partial_i = \frac{1-s_i}{x_i - x_{i+1}}$$

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- **2018** Assaf–S prove that Schubert polynomials are Demazure truncations of Stanley symmetric functions.

Demazure crystals

$X \subseteq B(\lambda)$, define $\mathfrak{D}_i$ as

$$\mathfrak{D}_i X = \{b \in B(\lambda) \mid f_i^k(b) \in X \text{ for some } k \geqslant 0\}$$

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Definition (Demazure crystal)

$w = s_{i_1} s_{i_2} \cdots s_{i_k} \in S_n$ reduced expression, $u_\lambda \in B(\lambda)$ highest weight vector

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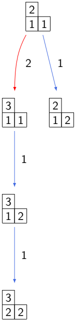
Theorem (Littelmann (conjectured), Kashiwara (proven) 1993)

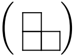
$\operatorname{ch} B_w(\lambda)$ *is a Demazure character.*

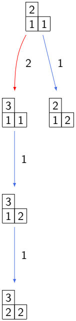
Demazure crystal

$$B_{s_1 s_2}\left(\ydiagram{1,2}\right)$$



Demazure crystal

$B_{s_1 s_2}$ 



Theorem (Assaf, S. 2018)

Schubert polynomials are Demazure truncations of Stanley symmetric functions.

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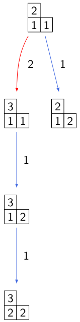

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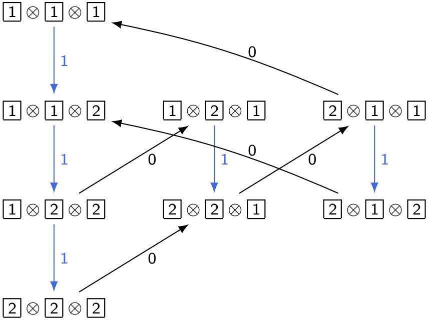
Reason 7

Crystals well behaved with respect to geometry and Demazure truncations.

Outline

Affine crystals



One dimensional configuration sums

Why affine crystals?

One dimensional configuration sums

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- energy function $E : B_N \otimes \cdots \otimes B_1 \to \mathbb{Z}$

$$E(e_i(b)) = E(b) \qquad \text{for } 1 \leqslant i \leqslant n$$
$$E(e_0(b)) = E(b) - 1$$

if e_0 does not act on leftmost step in $b = b_N \otimes \cdots \otimes b_1$.

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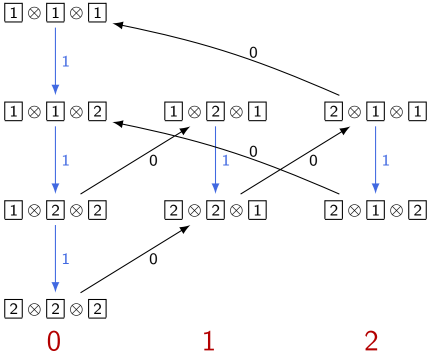
- characters of conformal field theories as limits of $X(\lambda, B)$

Energy function



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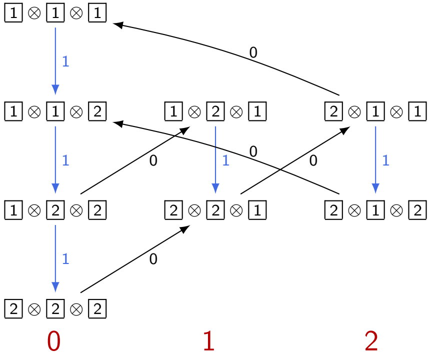
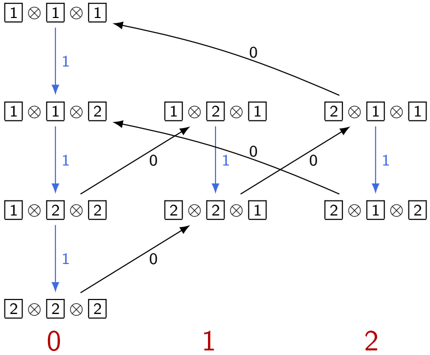


$$X((2,1),B) = 1 + q + q^2$$

Energy function



$X((2,1),B) = 1 + q + q^2$

Reason 8

Affine crystals give the energy function and one-dimensional configuration sums.

Kirillov–Reshetikhin crystals

Kirillov-Reshetikhin crystal $B^{r,s}$

$$B^{r,s} \cong B(s^r) \quad \text{as } \{1, 2, \ldots, n-1\}\text{-crystal}$$

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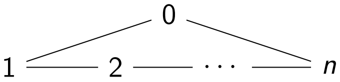

automorphism σ

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$$\begin{tikzcd}[column sep=small] & & 0 \arrow[dll, dash] \arrow[drr, dash] & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & \cdots \arrow[r, dash] & n \end{tikzcd}$$

automorphism σ

Promotion operator pr uniquely defined by Shimozono

$$\begin{tikzcd} B^{r,s} \arrow[r, "\mathrm{pr}"] \arrow[d, "f_a"'] & B^{r,s} \arrow[d, "f_{a+1}"] \\ B^{r,s} \arrow[r, "\mathrm{pr}"'] & B^{r,s} \end{tikzcd}$$

$$\langle \alpha_{a+1}^\vee \,, \mathrm{wt}(\mathrm{pr}(b)) \rangle = \langle \alpha_a^\vee \,, \mathrm{wt}(b) \rangle$$

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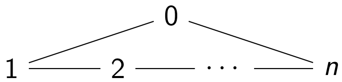

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\end{tikzcd}$$

$$\langle \alpha_{a+1}^\vee , \mathrm{wt}(\mathrm{pr}(b)) \rangle = \langle \alpha_a^\vee , \mathrm{wt}(b) \rangle$$

Then $\quad e_0 = \mathrm{pr}^{-1} \circ e_1 \circ \mathrm{pr} \quad f_0 = \mathrm{pr}^{-1} \circ f_1 \circ \mathrm{pr}$

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

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Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

3	3	4
2	2	3
$\bullet$	1	2

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•	•	2

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Example

$$\begin{array}{|c|c|c|}\hline 4 & 4 & 4 \\ \hline 3 & 3 & 3 \\ \hline 1 & 1 & 2 \\ \hline \end{array}$$

Promotion

Crystal commutor: (Henriquez, Kamnitzer 2006)

$$\begin{aligned} \sigma_{B,C} \colon B \otimes C &\to C \otimes B \\ b \otimes c &\mapsto \eta(\eta(c) \otimes \eta(b)) \end{aligned}$$

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Lusztig involution:

$$\eta \colon B \to B$$

η maps highest weight to lowest weight and maps e_i to $f_{i'}$ with $\omega_0(\alpha_i) = -\alpha_{i'}$

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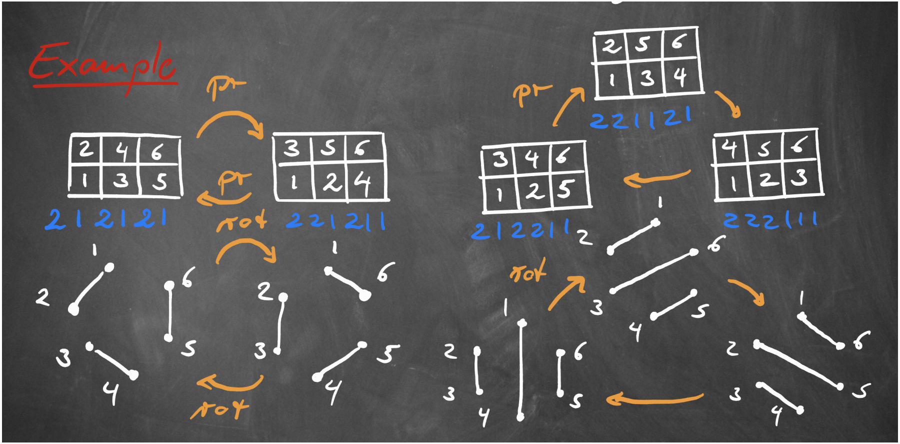
Definition (Promotion)

$u \in B^{\otimes n}$ highest weight

$$\mathrm{pr}(u) = \sigma_{C^{\otimes n-1},C}(u)$$

cyclic action on highest weight elements

Promotion – example



Cyclic sieving phenomenon

Theorem (Fontaine, Kamnitzer 2016, Westbury 2016, Pappe, Pfannerer, S., Simone 2023)

Highest weight elements in $B^{\otimes n}$ of weight zero, promotion, one-dimensional configuration sums gives rise to cyclic sieving phenomenon.

Cyclic sieving phenomenon

Theorem (Fontaine, Kamnitzer 2016, Westbury 2016, Pappe, Pfannerer, S., Simone 2023)

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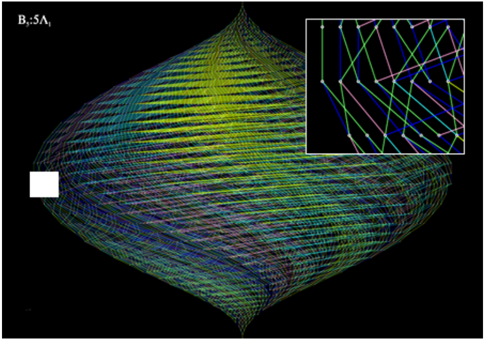
Cyclic sieving phenomenon: polynomials evaluated at roots of unity related to sizes of orbits under cyclic action

Reason 9

Crystals give rise to cyclic sieving phenomena and promotion gives a cyclic action.

Thank you !

Thank you !



Reason 10

Crystals are beautiful!