Math 127C Homework 1 (first two parts), Spring 2021

Due: Friday, May 12

First Part

(1) (Triangle Inequality) [Exercise 1.1.(b)] Prove that

$$\| \overrightarrow{x} + \overrightarrow{y} \| \leq \| \overrightarrow{x} \| + \| \overrightarrow{y} \|.$$

[*Hint:* Compute $\langle \overrightarrow{x} + \overrightarrow{y}, \overrightarrow{x} + \overrightarrow{y} \rangle$ and apply the Cauchy-Schwarz inequality which says that $\langle \overrightarrow{x}, \overrightarrow{y} \rangle \leq \| \overrightarrow{x} \| \| \overrightarrow{y} \|$.]

(2) (Matrix supremum norm)[Exercise 1.2] If A is an r by m matrix and B is an m by c matrix show that

$$|AB| \leq m|A||B|.$$

(3) (Theorem 18.3) Find a shortest sequence of type (2) and type (3) elementary row operations which have the effect of switching the first two rows of a matrix. Show that there is no such sequence using only type (2) operations.

(4) (Theorem 1.6) Prove that if B is the matrix obtained by applying an elementary row operation to A, then

$$\operatorname{rank} B = \operatorname{rank} A.$$

Second Part

(5) (Inverse via Row Operations)[Exercise 1.2.ab]

(a) Let A be an n by n matrix of rank n. By applying elementary row operations to A, one can reduce A to the identity matrix. Show that by applying the same operations in the same order to the identity matrix one obtains the matrix A^{-1}.

(b) Calculate A^{-1} by the above method if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

(6) (Block Determinant)[Exercise 1.2.6] Show that if

$$M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$$

is a block matrix for which A, D and M are square then $\det M = (\det A)(\det D)$.

Hint: First show that

$$\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C & D \end{bmatrix} = M.$$

Third Part

(7) (Inverse Stability)

(a) Consider the metric space $X = (\mathbb{R}^{r\times m+m\times c}, d_{|\cdot|})$ with the supremum metric. Elements of X are pairs of matrices (A, B) of sizes r by m and m by$c$. Consider the subspace $Y = \{(A, B)|AB \text{ is invertible}\} \subseteq X$.
Show that Y is open in X.
You may use the fact that the matrix multiplication map from X to $Z = (\mathbb{R}^{r\times c}, d_{|\cdot|})$ is continuous.

(b) Use (a) to show that there is some positive number ϵ so that if $A \in \mathbb{R}^{2\times 3}$ and $B \in \mathbb{R}^{3\times 2}$ with the entries of A differing from those of

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

by at most ϵ and the entries of B differing from those of

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

by at most ϵ then AB is an invertible two by two matrix.