1. Sol: By definition of f, we know:

$$f(w,x,y,z) = wy + xz$$
$$\leqslant |w|\cdot|y| + |x|\cdot|z|$$
$$\leqslant 2\cdot|(w,x,y,z)|^2 \qquad ①$$

Setting $\delta = \sqrt{\frac{\varepsilon}{2}}$ and $|(w,x,y,z)| < \delta$, by ①:

$$f(w,x,y,z) < 2\delta^2 = \varepsilon.$$

This completes the proof / construction of δ.

5. Sol:

(a) For any $(x,y) \neq (0,0)$, we know f is differentiable, and thus:

$$(Df)(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$$

$$\text{quotient rule:} = \left(y\cdot\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}, x\cdot\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\right)$$

$$= \frac{1}{(x^2+y^2)^2}\cdot\left(y\cdot(y^2-x^2), x\cdot(x^2-y^2)\right)$$

$$\Longrightarrow (Df)(1,2) = \frac{1}{(1^2+2^2)^2}\cdot\left(2\times(2^2-1^2), 1\times(1^2-2^2)\right) = \left(\frac{6}{25}, \frac{-3}{25}\right)$$

(b) By definition of directional derivative,

$$f'((0,0);(1,2)) = \lim_{t\to 0} \frac{f((0,0)+t(1,2))-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(t,2t)}{t}$$

$$= \lim_{t\to 0} \frac{1}{t}\cdot\left(\frac{2t^2}{t^2+4t^2}\right)$$

$$= \lim_{t\to 0} \frac{2}{5t}$$

Hence $f'((0,0);(1,2))$ does not exist.

(c) $$f'((0,0);(1,0)) = \lim_{t\to 0} \frac{f((0,0)+t(1,0))-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(t,0)}{t}$$

$$= 0$$

Similarly, $f'((0,0);(0,1)) = \lim_{t\to 0} \frac{f(0,t)}{t} = 0$

$\Longrightarrow$ $Jacf((0,0)) = (0,0)$.