Problem 1: Let $f:\mathbb{R}^4 \longrightarrow \mathbb{R}$, $(w,x,y,z) \longmapsto \langle (w,x),(y,z)\rangle$. $\forall \varepsilon>0$, Find $\delta>0$ s.t.

$$|(w,x,y,z)|<\delta \Longrightarrow |f(w,x,y,z)|<\varepsilon \quad (\text{i.e. show continuity @ } \vec{0}\in\mathbb{R}^4)$$

pf: Notice $|f(w,x,y,z)| = |\langle (w,x),(y,z)\rangle| = |wy+xz|$

Denote $|(w,x,y,z)| =: m$. $\therefore |wy+xz| \le 2m^2$

$\therefore$ Pick $\delta < \sqrt{\frac{\varepsilon}{2}} \Rightarrow m<\delta \Rightarrow 2m^2 < 2\delta^2 = \varepsilon$. $\square$.

Problem 2: find (a,b,c) s.t. $\begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{pmatrix}$

pf:

$$\begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

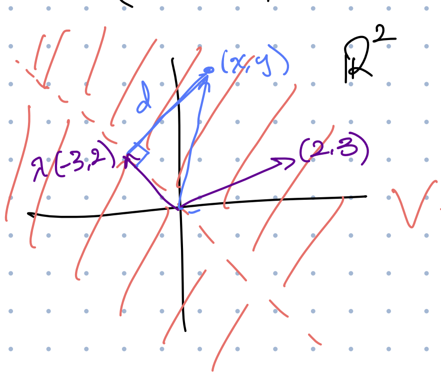
$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 5/2 & -1/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & 2/5 \\ 0 & 0 & 1 & -1/5 \end{pmatrix}$$

$$\boxed{\therefore (a,b,c) = (1/5,\ 2/5,\ -1/5).}$$

Problem 3:

a) $V := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \langle (a,b),(2,3)\rangle \neq 0 \right\}$ is open in $\mathbb{R}^2$



$V :=$ vectors not orthogonal to $(2,3)$

Known

$\lambda = \langle (x,y),(-3,2)\rangle = -3x+2y$.

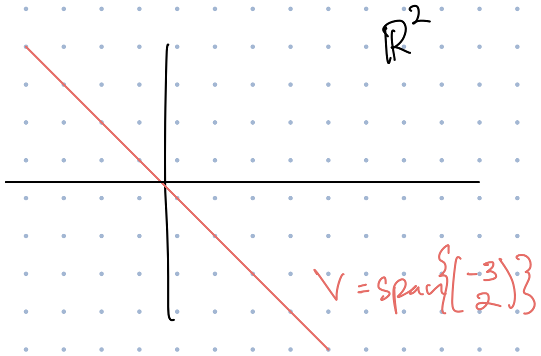
$\therefore \|\lambda(-3,2)\|^2 + d^2 = \|(x,y)\|^2$

$\therefore d = \sqrt{x^2+y^2+|\lambda|^2(9+4)}$

Take $\varepsilon < d \Rightarrow B_\varepsilon((x,y)) \subseteq V$.

$\Rightarrow V$ is open. $\square$.

b) $V := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \,\middle|\, \langle (a,b), (2,3) \rangle = 0 \right\}$ is closed in $\mathbb{R}^2$



pf: $f(a,b) = \langle (a,b), (2,3) \rangle = 2a + 3b$ is cts

$\therefore$ Since $V = f^{-1}(\{0\})$ and $\{0\}$ is closed in $\mathbb{R}^2$

V is also closed.

$\square$

Problem 4: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ continuous.

Assume for $|(x,y)| > 10$, that $f(x,y) = 0$.

Show $\exists$ some pt. in $\mathbb{R}^2$ which f takes it's maximum value.

pf: Consider $K := \{ (x,y) \in \mathbb{R}^2 \mid |(x,y)| \leq 10 \} \subseteq \mathbb{R}^2$.

K is closed + bded $\Longrightarrow$ by Heine-Borel K is compact in $\mathbb{R}^2$.

$\therefore$ Since $f : K \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is cts, By extreme value thm, f achieves a maximum on K, i.e. $\exists (x^*, y^*) \in K$ s.t. $f(x^*, y^*) \geq f(x,y)$ for all $(x,y) \in K$.

$\therefore$ maximum of f on $\mathbb{R}^2$ is: $\max f(x,y) = \max \{ f|_K, f|_{\mathbb{R}^2 \setminus K} \}$

$= \max \{ f(x^*, y^*), 0 \}$. $\square$.

Problem 5: Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ s.t. $f(x,y) := \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

This function is differentiable everywhere except $(0,0)$.

a) Find the 2×1 total derivative matrix $(Df)(1,2)$

pf: Since f is diffable @ $(1,2)$ (this is given)

$$Df(x,y) := \left(\frac{\partial f}{\partial x}(x,y) \quad \frac{\partial f}{\partial y}(x,y) \right) = \left(\frac{\partial}{\partial x}\left[\frac{xy}{x^2+y^2} \right] \quad \frac{\partial}{\partial y}\left[\frac{xy}{x^2+y^2} \right] \right)$$

$$= \left(y \left[\frac{y^2 - x^2}{(x^2+y^2)^2} \right] \quad x \left[\frac{x^2 - y^2}{(x^2+y^2)^2} \right] \right) = \frac{y^2 - x^2}{(x^2+y^2)^2} \begin{pmatrix} y & -x \end{pmatrix}$$

$$\therefore Df(1,2) = \frac{4-1}{(1+4)^2} \begin{pmatrix} 2 & -1 \end{pmatrix} = \frac{3}{25} \begin{pmatrix} 2 & -1 \end{pmatrix}.$$

$\square$

b) Find $f'((0,0);(1,2))$ or show it doesn't exist.

Note: f is not diff'able @ $(0,0)$ $\therefore$ need to use limit def.

pf: $$f'((0,0);(1,2)) := \lim_{t\to 0} \frac{f((0,0)+t(1,2)) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{2t^2}{t^2+4t^2}}{t} = \lim_{t\to 0} \frac{2}{5t} = \text{DNE}$$

$\square$

c) Find the 2×1 Jacobian matrix $\text{Jac}\, f((0,0))$

Note: $\text{Jac} f((0,0)) = \begin{pmatrix} D_1 f(0,0) & D_2 f(0,0) \end{pmatrix} = \begin{pmatrix} f'((0,0);(1,0)) & f'((0,0);(0,1)) \end{pmatrix}$

pf: $$f'((0,0);(1,0)) = \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t^2} = 0$$

$$f'((0,0);(0,1)) = \lim_{t\to 0} \frac{f((0,0)+t(0,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t^2} = 0.$$

$\therefore \text{Jac} f(0,0) = \begin{pmatrix} 0 & 0 \end{pmatrix}$. $\square$

Problem 6: Consider $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $f((x,y)) = (\underbrace{xy}_{f_1(x,y)}, \underbrace{x+y}_{f_2(x,y)})$

a) Compute all directional derivatives of f.

pf:

*Since $f_1(x,y), f_2(x,y)$ smooth $\mathbb{R}^2 \to \mathbb{R}$ (polynomials in x,y)

$$\Rightarrow Df(x,y) = \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix} = \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}$$

$$\therefore \boxed{f'((x,y);(h,k))} = Df(x,y)\cdot\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \boxed{\begin{pmatrix} hy + xk \\ h+k \end{pmatrix}}$$

b) Use a) to show f is ctsly differentiable.

pf: Notice in the 2×2 matrix $Df(x,y)$ containing all partials $D_j f_i = \frac{\partial f_i}{\partial x_i}$ each partial is either a constant or linear func. in x and y $\Rightarrow D_j f_i$ are cts $\forall i,j \in \{1,2\}$. $\square$

$x_1 := x$

$x_2 := y$

$$\boxed{\begin{aligned} D_1 f_1 &= \frac{\partial f_1}{\partial x} = y & D_2 f_1 &= \frac{\partial f_1}{\partial y} = x \\ D_1 f_2 &= \frac{\partial f_2}{\partial x} = 1 & D_2 f_2 &= \frac{\partial f_2}{\partial y} = 1. \end{aligned}}$$