1. Problems 200-2015-1: Metric Spaces

Problem 1.1. (Wi 02 4) If $X$ is a metric space prove or find a counterexample to:

1. If $X$ is compact, then it is complete.
2. If $X$ is complete, then it is compact.

Problem 1.2. (Wi 03 3) Prove that the space of all polynomials on $[0,1]$ which vanish at $0$ is not complete with the sup norm and find its completion.

Problem 1.3. (Wi 03 4a) Find the completion of $\mathbb{R}$ with the metric $d(x,y) = |\arctan(x) - \arctan(y)|$.

Problem 1.4. (Fa 03 3) Let $0 < r < 1$, $(X,d)$ a metric space and $T : X \to X$ with fixed point $p$ and every $d(Tx,Ty) \leq rd(x,y)$. Show that $d(x,p) \leq \frac{d(x,Tx)}{1-r}$. 

Problem 1.5. (Fa 04 6) Show that there is some number $e > 0$ such that for every $0 \leq E < e$ and $g$ continuous on $[0,1]$ there is a unique continuous function $f$ on $[0,1]$ with

$$f(x) = g(x) + E \int_0^1 (x-y)^2 f(y) dy + \frac{1}{2} \sin(f(x))$$

Problem 1.6. (Wi 08 1) Consider the sequence $\{f_n\}$ of polynomials with $f_n(x) = (-1)^n x^n (1-x)$.

1. Show that $\{\sum_0^\infty f_n\}$ converges uniformly in $[0,1]$.
2. Show that $\{\sum_0^\infty |f_n|\}$ converges pointwise but not uniformly in $[0,1]$.

Problem 1.7. (Wi 08 2) Write $e$ for the Euclidean metric on $\mathbb{R}^2$ and define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ with $d(u,v) = e(u,v)$ if there is some $c \in \mathbb{R}$ with $cu = v$ and $d(u,v) = e(u,0) + e(0,v)$ otherwise. Show that $(\mathbb{R}^2,d)$ is a metric space and find $U \subseteq \mathbb{R}^2$ which is open in $(\mathbb{R}^2,d)$ but not open in $(\mathbb{R}^2,e)$.

Problem 1.8. (Fa 08 2) Show that if $\{f_n\}$ is a sequence of continuously differentiable functions on $[0,1]$ and both the original sequence and the sequence of derivatives are uniformly uniformly bounded then there is a uniformly convergent subsequence.

Problem 1.9. (Fa 08 4) Show that if $X$ is a compact metric space then the set of functions $f(x)g(y)$ with $f,g \in C(X)$ generates a dense subalgebra of $C(X^2)$.

1.1. Demonstration Problems

Problem 1.10. (Wi 03 4b) Find the completion of $\mathbb{R}$ with the metric $d(x,y) = |e^x - e^y|$.

Problem 1.11. Find the uniform closure of the even degree polynomials in $C(-1,1)$.

Problem 1.12. Show that if $T(f(t)) = t^2 + \frac{1}{3} \int_0^1 sf^2(s)ds$ there is $u \in C(0,1)$ with $\{T^n f\}$ converging uniformly to $u$ for any choice of $f \in C(0,1)$. 

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