
Problem 5.1. (Fa 03 5)(Wi 05 4) Find the point spectra of $R$ and $L$ the right and left shift operators on $\ell^2(\mathbb{N})$. (That is $R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$ and $L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$).

Problem 5.2. (Wi 04 6) Let $T \in B(H)$ be a self-adjoint, compact operator on a Hilbert space for which $\frac{1}{4}T^3 - T^2 + \frac{1}{4}T = 0$.

1. Show that $T = P + Q$ with $P$ and $Q$ orthogonal projections.
2. Show that these projections ($P$ and $Q$) must be finite rank.

Problem 5.3. (Wi 07 6) Show that the spectrum of every unitary operator lies on the unit circle.

Problem 5.4. (Wi 08 4) Assume that $H$ is a complex separable Hilbert space and $A \in B(H)$ with spectrum $\sigma$, resolvent set $\rho$ and resolvent operators $R(\mu) = (\mu I - A)^{-1}$.

1. Show that if $\mu \in \rho$ and $|\nu - \mu| < \|R(\mu)\|^{-1}$ then $R(\nu) = [I - (\mu - \nu)R(\mu)]^{-1}R(\mu)$.
2. Show that if $\mu \in \rho$ then $\|R(\mu)\| \geq (d(\mu, \sigma))^{-1}$.

Problem 5.5. (Wi 09 4b) Find and classify the spectrum of $T$ if $T \in B(H)$ with $H$ a complex Hilbert space, $-T = T^*$, $T^2 = -I$ and $T \neq \pm iI$.

Problem 5.6. (Fa 11 2) Show that if $T \in B(X)$ is a bounded linear operator on a Banach space $X$ then

1. $T^* \in B(X^*)$
2. $\|T^*\| \leq \|T\|$
3. Show that if $\|T\|$ is an eigenvalue for $T$ it is also an eigenvalue for $T^*$.
   Hint: Consider the Cesaro means $\psi_N = \sum_{n=1}^{N} T^n \phi$.

Problem 5.7. (Fa 12 3)

1. Show that if a linear operator on a Hilbert space has norm at most one then it has the same fixed points as its adjoint.
2. Assume that $\lambda$ is an eigenvalue of $T \in B(H)$ with $H$ complex Hilbert.
   (a) Is it true that $\overline{\lambda}$ must be an eigenvalue of $T^*$?
   (b) Is it true that $\overline{\lambda}$ must be in the spectrum of $T^*$?

Problem 5.8. (Fa 12 2) Assume that $k \in C_0[0, 1]$ and define $T \in B(L^2[0, 1])$ by setting $(Tf)(x) = k(x) \int_0^1 k(t)f(t)dt$ for any $f \in L^2[0, 1]$.

1. Show that $T$ is self-adjoint.
2. Show that $T^2 = rT$ for some number $r$.
3. Find the spectral radius of $T$.

Problem 5.9. (Sp 13 2) Show that if $\{T_i\} \in B(X)$ is a sequence of bounded operators on a Banach space $X$ which all have the same spectrum $\sigma = \sigma(T_i)$ and converge in norm to $T$ then $\sigma \subseteq \sigma(T)$. 