Problem 7.1. (Wi 08 5) For each $1 \leq p < \infty$ find the values of $a$ for which $|x|^a \in W^{1,p}(-1,1)$.

Problem 7.2. (Fa 08 3) Let $V_f$ be the closed subspace of $L^1(\mathbb{R})$ generated by the translates of $f$.

(1) Show that if $\hat{f}(z) = 0$ and $h \in V_f$ then $\hat{h}(z) = 0$.

(2) Show that if $V_f = L^1(\mathbb{R})$ then $\hat{f}$ is nowhere 0.

Problem 7.3. (Wi 09 5) Consider the linear operator $H : S(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ defined by $(Hf)(z) = i \frac{d}{dz} \hat{f}(z)$ where $S$ is the Schwartz space.

(1) Why is $Hf \in L^2(\mathbb{R})$?

(2) Show that if $f \in S(\mathbb{R})$ and $Hf \in L^1(\mathbb{R})$ then $\int f(x) dx = 0$.

Problem 7.4. (Fa 09 1) Let $\{\eta_r\}_{r>0}$ denote the family of standard mollifiers on $\mathbb{R}^2$ and for each $u \in L^2(\mathbb{R}^2)$ define $u_r = \eta_r \ast u$. Show that there is a constant $C$ so that for every $u \in L^2(\mathbb{R}^2)$ there is $C\|Du_r\|_2 \leq \|u\|_2$.

Problem 7.5. (Fa 09 2) Show that $\log |x| \in H^1(\mathbb{B})$ if $\mathcal{B}$ is the unit ball in $\mathbb{R}^3$.

Problem 7.6. (Sp 10 5) Show that for every $r < s < t$ and $e > 0$ there is $C$ so that for every $u \in H^1(\mathbb{R})$ one has $\|u\|_{s} \leq e\|u\|_{t} + C\|u\|_{r}$. Here $\|u\|_{s} = \int_{\mathbb{R}} (1 + |z|^2)^s |\hat{u}(z)|^2 dz$ is the Sobolev norm on $H^s(\mathbb{R})$.

Problem 7.7. *(Fa 10 4) Show that if $g(x) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}} \in L^1(\mathbb{R}^3)$ and $G(f) = g \ast f$ is the convolution then $G \in B(L^2(\mathbb{R}^3))$ with $\|G\|_{op} \leq 1$.

Problem 7.8. (Fa 12 4) Consider the heat kernels $H_t(x) = (4\pi t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4t}}$ on $\mathbb{R}^3$. Show that if $u \in L^3(\mathbb{R}^3)$ then $t^\frac{1}{2}\|H_t \ast u\|_\infty \rightarrow 0$ as $t \rightarrow 0^+$.

Problem 7.9. (Fa 12 6) Show that if $\int_{\mathbb{R}} |\hat{f}(k)|^2 (1 + |k|^2)^s dk < \infty$ for some $s > \frac{3}{2}$ then $f$ is globally Lipschitz. Recall that $f$ is globally Lipschitz if there is a constant $K$ so that every $x$ and $y$ have $|f(x) - f(y)| \leq K|x - y|$.

Problem 7.10. (Sp 13 6) Show that for every $1 \leq p \leq q$ there is $C$ so that for every $u \in W_0^{1,p}(0,1)$ there is $\|u\|_{L^q} \leq C\|u\|_{W^{1,p}}$.

Hint: First show that $\|u\|_{L^\infty} \leq C\|u\|_{W^{1,p}}$. 