MAT200, 2021, Day 4, Banach Spaces

(1) Fall 2020, Problem 2 Let $X$ be a normed linear space and $Y$ be a Banach space. Let $K(X, Y)$ be the set of all compact operators from $X$ to $Y$. Show that $K(X, Y)$ is a closed subspace of the space $B(X, Y)$ of all bounded linear operators from $X$ to $Y$. (You may assume that $K(X, Y)$ is a subspace of $B(X, Y)$.)

(2) Fall 2020, Problem 4 Let $X = C([0, 1])$ with the norm $\| \cdot \|_\infty$. For any $f \in C([0, 1])$, let

$$
\Phi(f) = \int_0^{1/2} f(x)dx - \int_{1/2}^{1} f(x)dx.
$$

Show that $\Phi$ is a bounded linear functional on $X$ with $\| \Phi \| = 1$. Moreover, show that $\Phi$ does not attain its norm in $X$. (i.e., there is no $f \in X$ with $\| f \|_\infty = 1$ such that $| \Phi(f) | = \| \Phi \|$).

(3) Fall 2017, Problem 2 Let $X$ be a Banach space with dual space $X^*$ and let $A \subseteq X$ be a linear subspace. Define the annihilator $A^\perp \subseteq X^*$ of $A$ by

$$
A^\perp = \{ f \in X^* : f(x) = 0 \text{ for all } x \in A \}.
$$

Prove that $A$ is dense in $X$ if and only if $A^\perp = 0$.

(4) Fall 2016, Problem 5 Let $K : [0, 1] \times [0, 1] \to \mathbb{R}$ be a continuous function and fix $1 < p < \infty$. Given $f \in L^p([0, 1])$, define $Tf : [0, 1] \to \mathbb{R}$ by

$$
Tf(x) = \int_0^1 K(x, y)f(y)dy.
$$

(a) Prove that $Tf$ is a continuous function.

(b) Prove that the image under $T$ of the unit ball in $L^p([0, 1])$ is precompact in $C([0, 1])$.

(5) Fall 2015, Problem 2 Let $T$ be a linear operator on a Banach space. Show that $T$ is bounded if and only if it is continuous.

(6) Fall 2015, Problem 5 Fix a continuous function $f : [0, 1] \to \mathbb{R}$ Consider the multiplication operator $M_f : C^0([0, 1]) \to C^0([0, 1])$ on the space $C^0([0, 1])$ of continuous functions on $[0, 1]$ defined by $[M_f(g)](x) = f(x)g(x)$ for all $x \in [0, 1]$ and $g \in C^0([0, 1])$. Calculate $\| M_f \|$ and show that $M_f$ is a compact operator if and only if $f = 0$.

(7) Fall 2011, Problem 3 Let $H$ be a complex Hilbert space and denote by $B(H)$ the Banach space of all bounded linear transformations (operators) of $H$ considered with the operator norm.

(a) What does it mean for $A \in B(H)$ to be compact? Give a definition of compactness of an operator $A$ in terms of properties of the image of bounded sets, e.g., the set $\{ Ax \in H, \| x \| \leq 1 \}$.

(b) Suppose $H$ is separable and let $\{ e_n \}_{n \geq 0}$ be an orthonormal basis of $H$. For $n \geq 0$, let $P_n$ denote the orthogonal projection onto the subspace spanned by $e_0, \ldots, e_n$. Prove that $A \in B(H)$ is compact iff the sequence $(P_nA)_{n \geq 0}$ converges to $A$ in norm.
Fall 2010, Problem 5 Consider the map which associates to each sequence 
\(\{x_n : n \in \mathbb{N}, x_n \in \mathbb{R}\}\) the sequence, \(\{(F(\{x_n\}))_m \in \mathbb{R}\}\), defined as follows:

\[ F(\{x_n\})_m := \frac{x_m}{m} \quad \text{for} \quad m = 1, 2, \ldots \]

(a) Determine (with proof) the values of \(p \in [1, \infty]\) for which the map 
\(F : \ell^p \to \ell^1\) is well-defined and continuous.

(b) Next, determine the values of \(q \in [1, \infty]\) for which the map \(F : \ell^q \to \ell^2\) 
is well-defined and continuous.