1. Let \((X, d)\) be a metric space. Suppose that \(\{x_n\}_{n=1}^\infty \subset X\) is a sequence and set \(\varepsilon_n := d(x_n, x_{n+1})\). Show that for \(m > n\)

\[
d(x_n, x_m) \leq \sum_{k=n}^{m-1} \varepsilon_k \leq \sum_{k=n}^{\infty} \varepsilon_k.
\]

Conclude from this that if

\[
\sum_{k=1}^{\infty} \varepsilon_k = \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty
\]

then \(\{x_n\}_{n=1}^\infty\) is Cauchy. Moreover, show that if \(\{x_n\}_{n=1}^\infty\) is a convergent sequence and \(x = \lim_{n\to\infty} x_n\) then

\[
d(x, x_n) \leq \sum_{k=n}^{\infty} \varepsilon_k.
\]

2. Show that limits of Cauchy sequences in a metric space are unique.

3. Let \((X, d)\) be a metric space and suppose that \(\{x_n\}_{n=1}^\infty \subset X\) converges to \(x \in X\). Show that \(\{x_n\}_{n=1}^\infty\) is a Cauchy sequence. Give an example where the converse is not true.

4. Let \((X, d)\) be a complete metric space. Let \(A \subset X\) be a subset of \(X\) viewed as a metric space using \(d|_A\). Show that \((A, d|_A)\) is complete iff \(A\) is a closed subset of \(X\).

5. Let \(\{a_n\}_{n=1}^\infty\) be a sequence of real numbers. Then

(a) \(\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n\), and \(\lim_{n\to\infty} a_n\) exists in \(\bar{\mathbb{R}}\) iff

\[
\lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} \sup a_n \in \bar{\mathbb{R}}.
\]

(b) There is a subsequence \(\{a_{n_k}\}_{k=1}^\infty\) of \(\{a_n\}_{n=1}^\infty\) such that

\[
\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n.
\]
6. Show that $V \subset X$ is open iff for every $x \in V$ there is a $\delta > 0$ such that $B_{\delta}(x) \subset V$. In particular show $B_{\delta}(x)$ is open for all $x \in X$ and $\delta > 0$. Recall that by definition $V$ is not open iff $V^c$ is not closed.

Note: for this problem you must use the definition of open set from class on Monday (HN defines open sets differently) which is:

A set $V \subset X$, a metric space, is open if $V^c$ is closed. A set $F \subset X$ is closed iff every convergent sequence $\{x_n\}_{n=1}^{\infty}$ has a limit point in $x \in F$.

7. Suppose that $(X, \rho)$ and $(Y, d)$ are metric spaces and $A$ is a dense subset of $X$.

(a) Show that if $F : X \to Y$ and $G : X \to Y$ are two continuous functions such that $F = G$ on $A$ the $F = G$ on $X$. Hint: Consider the set $C := \{x \in X : F(x) = G(x)\}$.

(b) Suppose $f : A \to Y$ is a function which is uniformly continuous, i.e. for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(f(a), f(b)) < \varepsilon \text{ for all } a, b \in A \text{ with } \rho(a, b) < \delta.$$

Show that there is a unique continuous function $F : X \to Y$ such that $F = f$ on $A$. Hint: each point $x \in X$ is a limit of a sequence consisting of elements from $A$.

(c) Let $X = \mathbb{R} = Y$ and $A = \mathbb{Q} \subset X$, find a function $f : \mathbb{Q} \to \mathbb{R}$ which is continuous on $\mathbb{Q}$ but does not extend to a continuous function on $\mathbb{R}$. 