This is an *optional* practice midterm. It has been designed with intention to be longer and more difficult than what should be encountered on the actual midterm. However, I have not seen the actual midterm, and have had no hand in designing it. Therefore, I will try to cover all bases in this practice midterm to adequately prepare you.

This second midterm should cover what you have learned from Chapter 3, which is roughly where you left off before the last midterm (although you did encounter a problem of conceptualizing the derivative on Midterm I). So the material you should be comfortable with include the power, product, quotient, and chain rules, as well as knowing the derivative of $e^x$ and $\ln(x)$, you trig. and inverse trig. derivatives, implicit differentiation, related rates, and linearization. You will notice that only one related rates problem is included in this practice test, and nothing on linearization, so consult the homework exercises to prepare more for these types of problems.

Unlike with my previous practice midterm, I am including a set of solutions as well, should you get stuck along the way or wish to check your answers. Note that, although I tried to work slowly through the solutions, I may be prone to error, or may have gone about calculating my answers differently than the way you have; feel free to email me if you think I made a mistake (jrsclark@math...).

Just as before, I encourage you to meet with the TA’s during their office hours, bringing any and all questions you have. You are in college now: you will become very accustomed to office hours over the next four years, so take advantage of them. As always, good luck, and may the force be with you.

**Exercise 1.** (Trig. Derivatives) Recall the derivatives of each of the following:

- \[ \frac{d}{dx} \sin(x) \]
- \[ \frac{d}{dx} \cos(x) \]
- \[ \frac{d}{dx} \tan(x) \]
- \[ \frac{d}{dx} \csc(x) \]
- \[ \frac{d}{dx} \sec(x) \]
- \[ \frac{d}{dx} \cot(x) \]
- \[ \frac{d}{dx} \sin^{-1}(x) \]
- \[ \frac{d}{dx} \cos^{-1}(x) \]
- \[ \frac{d}{dx} \tan^{-1}(x) \]
- \[ \frac{d}{dx} \csc^{-1}(x) \]
- \[ \frac{d}{dx} \sec^{-1}(x) \]
- \[ \frac{d}{dx} \cot^{-1}(x) \]

**Exercise 2.** (Basic Derivatives) Calculate the following derivatives.

- \[ (a) \frac{d}{dx} \ln(x) \]
- \[ (b) \frac{d}{dx} \ln(2) \]
- \[ (c) \frac{d}{dx} \log_2(x) \]
- \[ (d) \frac{d}{dx} \left( \frac{\sin(x)}{x} \right) \]
- \[ (e) \frac{d}{dx} \left( xe^{2x} - \frac{4}{x} \right)^7 \]
- \[ (f) \frac{d}{dx} \left( \sqrt[4]{x^4 - \frac{\tan(x)}{x}} \right) \]
Exercise 3. (Implicit Differentiation.) Calculate $\frac{dy}{dx}$ for the following equations. If the equation starts out as $y = f(x)$, then solve for $\frac{dy}{dx}$ strictly in terms of $x$; otherwise, your answer can involve both $x$ and $y$.

(a) $y = x^x$

(b) $y = (\sin(x))^x$

(c) $y = (\sin(x))^\cos(x)$

(d) $xy^2 = x + y$

(e) $xy = \tan^{-1}(x - y)$

(f) $y = \frac{\sqrt{3x+7(x^2-3)^7}}{\sqrt{2x-22(x-9)^4}}$

Exercise 4. (Related Rates) A 10 ft. ladder is initially leaning on a wall when the base of the ladder begins to slide away from the wall. When the highest point of the ladder is 6 ft. above the ground, the base of the ladder is moving at a rate of 2 ft/sec.

(a) Draw a picture of what’s happening above at the given instance. Create variable names: $x$ for the length of the base, $y$ for the height of the ladder off the ground, and $s$ for the length of the actual ladder. Label all things known, including rates of change, and put question marks for unknown values.

(b) Find the length of each side of the triangle described. Using this, find the rate of change $\frac{dy}{dt}$ at the given time in this instance.

(c) Find the rate of change $\frac{dA}{dt}$ of area of the triangle in the given instance.

(d) Find the rate of change $\frac{d\theta}{dt}$ of the angle between the ground and the ladder in the given instance.
Solutions.

Exercise 1. For the first 6, one thing you should keep in mind is that the derivative of a co-function is negative, whereas the derivative of a non co-function does not have a negative sign, e.g. \( \frac{d}{dx} \tan(x) = \sec^2(x) \) and \( \frac{d}{dx} \cot(x) = -\csc^2(x) \).

\[
\begin{align*}
\frac{d}{dx} \sin(x) &= \cos(x), & \frac{d}{dx} \cos(x) &= -\sin(x) \\
\frac{d}{dx} \tan(x) &= \sec^2(x), & \frac{d}{dx} \cot(x) &= -\csc^2(x) \\
\frac{d}{dx} \sec(x) &= \sec(x) \tan(x), & \frac{d}{dx} \csc(x) &= -\csc(x) \cot(x)
\end{align*}
\]

For the second 6, notice that \( \frac{d}{dx} \co \square = -\frac{d}{dx} \square \).

\[
\begin{align*}
\frac{d}{dx} \sin^{-1}(x) &= \frac{1}{\sqrt{1-x^2}}, & \frac{d}{dx} \cos^{-1}(x) &= -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx} \tan^{-1}(x) &= \frac{1}{1+x^2}, & \frac{d}{dx} \cot^{-1}(x) &= -\frac{1}{1+x^2} \\
\frac{d}{dx} \sec^{-1}(x) &= \frac{1}{|x|\sqrt{x^2-1}}, & \frac{d}{dx} \csc^{-1}(x) &= -\frac{1}{|x|\sqrt{x^2-1}}
\end{align*}
\]

Exercise 2.

(a) \( \frac{d}{dx} \ln(x) = \frac{1}{x} \).

(b) \( \ln(2) \) is a constant, so \( \frac{d}{dx} \ln(2) = 0 \) (since the derivative of any constant is just 0).

(c) Recall that we can always rewrite \( \log_B(x) = \frac{\ln(x)}{\ln(B)} \), so

\[
\frac{d}{dx} \log_2(x) = \frac{d}{dx} \frac{\ln(x)}{\ln(2)} = \frac{1}{\ln(2)} \cdot \frac{d}{dx} \ln(x) = \frac{1}{x \ln(2)}.
\]

(d) We use the quotient rule \( \frac{d}{dx} \left( \frac{f}{g} \right) = \frac{f'g - fg'}{g^2} \), where \( f'(x) = \cos(x) \) and \( (x)' = 1 \):

\[
\frac{d}{dx} \left( \frac{x \sin(x)}{x} \right) = \frac{(\cos(x))(x) - (\sin(x))(1)}{(x)^2} = \frac{x \cos(x) - \sin(x)}{x^2}.
\]

(e) Rewrite \( \frac{4}{x} \) as \( 4x^{-1} \). This will be an application of the chain rule a couple of times.

When encountering these problems, work slowly and neatly, for your own benefit (as well as for us graders):

\[
\begin{align*}
\frac{d}{dx} (xe^{2x} - 4x^{-1})^7 &= 7(xe^{2x} - 4x^{-1})^6 \cdot \frac{d}{dx} (xe^{2x} - 4x^{-1}) \\
&= 7(xe^{2x} - 4x^{-1})^6 \cdot (\frac{d}{dx} xe^{2x} - \frac{d}{dx} 4x^{-1}) \\
&= 7(xe^{2x} - 4x^{-1})^6 \cdot \left( [(1)e^{2x} + (x)(2e^{2x})] - [4(-1)x^{-2}] \right) \\
&= 7(xe^{2x} - 4x^{-1})^6 \cdot (e^{2x}[1 + 2x] + \frac{4}{x^2}).
\end{align*}
\]
(f) Rewrite $\sqrt{\Box} = \Box^{\frac{1}{2}}$:

$$\frac{d}{dx} \left( x^4 - \frac{\tan(x)}{x} \right)^{\frac{1}{4}} = \frac{1}{4} \left( x^4 - \frac{\tan(x)}{x} \right)^{-\frac{3}{4}} \cdot \frac{d}{dx} \left( \frac{x^4 - \tan(x)}{x} \right)$$

$$= \frac{1}{4} \left( x^4 - \frac{\tan(x)}{x} \right)^{-\frac{3}{4}} \cdot \left( 4x^3 - \frac{(sec^2(x))(x) - (\tan(x))(1)}{x^2} \right)$$

$$= \frac{1}{4} \left( x^4 - \frac{\tan(x)}{x} \right)^{-\frac{3}{4}} \cdot \left( 4x^3 - \frac{x \ sec^2(x) - \tan(x)}{x^2} \right).$$

**Exercise 3.** Recall the following list of rules about logs:

(1) $\log(A^B) = B \cdot \log(A)$

(2) $\log(A \cdot B) = \log(A) + \log(B)$

(3) $\log(A/B) = \log(A) - \log(B)$

(note: these rules holds for any base, not just base 10 or base e.)

(a) $y = x^x \Rightarrow \ln(y) = \ln(x^x) = x \cdot \ln(x)$. Take the derivative of both sides to get

$$\frac{1}{y} \frac{dy}{dx} = (1)(\ln(x)) + (x)(\frac{1}{x})$$

$$= \ln(x) + 1$$

$$\Rightarrow \frac{dy}{dx} = y \cdot (\ln(x) + 1)$$

$$= x^x(\ln(x) + 1).$$

(b) $y = \sin(x)^x \Rightarrow \ln(y) = \ln(\sin(x)^x) = x \cdot \ln(\sin(x))$, so

$$\frac{1}{y} \frac{dy}{dx} = (1) \cdot \ln(\sin(x)) + x \cdot \left( \frac{1}{\sin(x)} \cdot \cos(x) \right)$$

$$= \ln(\sin(x)) + x \cot(x)$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \left( \ln(\sin(x)) + x \cot(x) \right) = \sin(x)^x \left( \ln(\sin(x)) + x \cot(x) \right).$$

(c) $y = \sin(x)^{\cos(x)} \Rightarrow \frac{1}{y} \frac{dy}{dx} = \cos(x) \cdot \ln(\sin(x))$:

$$\frac{1}{y} \frac{dy}{dx} = [- \sin(x)] \cdot \ln(\sin(x)) + \cos(x) \cdot \left[ \frac{1}{\sin(x)} \cdot \cos(x) \right]$$

$$= \frac{\cos^2(x)}{\sin(x)} - \sin(x) \cdot \ln(\sin(x))$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \left( \frac{\cos^2(x)}{\sin(x)} - \sin(x) \cdot \ln(\sin(x)) \right)$$

$$= \sin(x)^{\cos(x)} \cdot \left( \frac{\cos^2(x)}{\sin(x)} - \sin(x) \cdot \ln(\sin(x)) \right).$$
(d) \[ xy^2 = x + y \]
\[ \Rightarrow (1)y^2 + x(2y \frac{dy}{dx}) = 1 + (1 \frac{dy}{dx}) \]
\[ \Rightarrow y^2 + 2xy \frac{dy}{dx} = 1 + \frac{dy}{dx} \]
\[ \Rightarrow 2xy \frac{dy}{dx} - \frac{dy}{dx} = 1 - y^2 \]
\[ \Rightarrow (2xy - 1) \frac{dy}{dx} = 1 - y^2 \]
\[ \Rightarrow \frac{dy}{dx} = \frac{1 - y^2}{2xy - 1}. \]

(e) \[ xy = \tan^{-1}(x - y) \]
\[ \Rightarrow (1)y + x(\frac{dy}{dx}) = \frac{1}{1 + (x - y)^2}(1 - \frac{dy}{dx}) \]
\[ \Rightarrow y + x \frac{dy}{dx} = \frac{1}{1 + (x - y)^2} - \frac{1}{1 + (x - y)^2} \frac{dy}{dx} \]
\[ \Rightarrow x \frac{dy}{dx} + \frac{1}{1 + (x - y)^2} \frac{dy}{dx} = \frac{1}{1 + (x - y)^2} - y \]
\[ \Rightarrow (x + \frac{1}{1 + (x - y)^2}) \frac{dy}{dx} = \frac{1}{1 + (x - y)^2} - y \]
\[ \Rightarrow \frac{dy}{dx} = \frac{\frac{1}{1+(x-y)^2} - y}{x + \frac{1}{1+(x-y)^2}}. \]

(f) The purposes of this exercise is to avoid the quotient rule altogether, as using it would take far too long. Instead, the three rules of logs helps make the problem much easier to solve.

\[ y = \frac{\sqrt{3x + 7(x^2 - 3)^7}}{\sqrt[7]{x - 22(x - 9)^4}} \]
\[ \Rightarrow \ln(y) = \ln\left(\frac{\sqrt{3x + 7(x^2 - 3)^7}}{\sqrt[7]{x - 22(x - 9)^4}}\right) \]
\[ = \ln(\sqrt{3x + 7(x^2 - 3)^7}) - \ln(\sqrt[7]{x - 22(x - 9)^4}) \]
\[ = \ln(\sqrt{3x + 7}) + \ln((x^2 + 3)^7) - \ln(\sqrt[7]{x - 22}) - \ln((x - 9)^4) \]
\[ = \frac{1}{2} \ln(3x + 7) + 7 \ln(x^2 + 3) - \frac{1}{3} \ln(7x - 22) - 4 \ln(x - 9), \]
so taking derivatives of both sides, we see
\[
\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left( \frac{1}{3x + 7} (3) + \frac{7}{x^2 + 3} (2x) - \frac{1}{3} \right) \frac{1}{7x - 22} (7) - \frac{4}{x} \left( \frac{1}{x - 9} \right) \\
= \frac{3}{2(3x + 7)} + \frac{14x}{x^2 + 3} - \frac{7}{3(7x - 22)} - \frac{4}{x - 9} \\
\Rightarrow \frac{dy}{dx} = y \cdot \left( \frac{3}{2(3x + 7)} + \frac{14x}{x^2 + 3} - \frac{7}{3(7x - 22)} - \frac{4}{x - 9} \right) \\
= \frac{\sqrt{3x + 7}(x^2 - 3)^{\frac{7}{2}}}{\sqrt{7x - 22}(x - 9)^4} \cdot \left( \frac{3}{2(3x + 7)} + \frac{14x}{x^2 + 3} - \frac{7}{3(7x - 22)} - \frac{4}{x - 9} \right). 
\]

**Exercise 5.**

(a) Above.

(b) The Pythagorean theorem \( s^2 = x^2 + y^2 \) tells us that \( x \) is 8 ft. in this instance. If we use implicit differentiation on both sides, we get \( 2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \). Now, we know \( \frac{ds}{dt} = 0 \) (because the length \( s \) is always 10; i.e. the overall length of the ladder is constant), so
\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0. 
\]
We know \( x, y, \) and \( \frac{dx}{dt}, \) so
\[
2(8)(2) + 2(6) \frac{dy}{dt} = 0 \\
\Rightarrow \frac{dy}{dt} = -\frac{8}{3} \frac{\text{ft}}{\text{sec}}. 
\]
(c) We use the standard area formula \( A = \frac{1}{2}xy \), so

\[
A = \frac{1}{2}xy
\]

\[
\Rightarrow \frac{dA}{dt} = \frac{1}{2}(\frac{dx}{dt})y + \frac{1}{2}x(\frac{dy}{dt})
\]

\[
= \frac{1}{2}(x\frac{dy}{dt} + y\frac{dx}{dt}).
\]

We now know all of \( x, y, \frac{dx}{dt} \) and \( \frac{dy}{dt} \), so plugging in we get

\[
\frac{dA}{dt} = \frac{1}{2}\left((8)(-\frac{8}{3}) + (6)(2)\right)
\]

\[
= -\frac{14}{3} \text{ ft}^2/\text{sec}.
\]

(d) There are several equivalent ways one can approach this, either by using \( \sin(\theta) = \frac{y}{s} \), \( \cos(\theta) = \frac{x}{s} \), \( \tan(\theta) = \frac{y}{x} \), or one of sec, csc, or cot. I will do two different ways here, just to illustrate how they will come to the same thing.

First method: Write \( \sin(\theta) = \frac{y}{s} = \frac{y}{10} \) (since \( s = 10 \) is constant). Then

\[
\sin(\theta) = \frac{y}{10}
\]

\[
\Rightarrow \cos(\theta) \frac{d\theta}{dt} = \frac{1}{10} \frac{dy}{dt}
\]

\[
\Rightarrow \frac{d\theta}{dt} = \frac{1}{10 \cos(\theta)} \frac{dy}{dt}
\]

\[
= \frac{1}{10(\frac{y}{s})}(-\frac{8}{3})
\]

\[
= -\frac{1}{3}.
\]

Second Method: Write \( \theta = \tan^{-1}(\frac{y}{x}) \), so

\[
\frac{d\theta}{dt} = \frac{1}{1 + (\frac{y}{x})^2} \left(\frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} \frac{dx}{dt}\right)
\]

\[
= \frac{1}{1 + (\frac{8}{8})^2} \left(\frac{1}{8}(-\frac{8}{3}) - \frac{6}{8^2}(2)\right)
\]

\[
= -\frac{1}{3}.
\]