Math 21C
Kouba
Absolute Convergence Test

**Absolute Convergence Test**: Consider the series \( \sum_{n=1}^{\infty} a_n \), which has both positive and negative terms. If \( \sum_{n=1}^{\infty} |a_n| \) converges (\( < \infty \)), then \( \sum_{n=1}^{\infty} a_n \) converges.

**Proof**: Consider that for \( n = 1, 2, 3, 4, \ldots \)

\[
a_n + |a_n| = \begin{cases} 
2 \cdot |a_n| , & \text{if } a_n > 0 \\
0 , & \text{if } a_n < 0 .
\end{cases}
\]

Thus,

\[
0 \leq a_n + |a_n| \leq 2 \cdot |a_n|
\]

for \( n = 1, 2, 3, 4, \ldots \). But the series \( \sum_{n=1}^{\infty} 2 \cdot |a_n| = 2 \cdot \sum_{n=1}^{\infty} |a_n| \) converges (Since scalar multiples of convergent series are convergent.), so that \( \sum_{n=1}^{\infty} (a_n + |a_n|) \) converges by the Comparison Test. We also have that \( \sum_{n=1}^{\infty} -|a_n| \) converges (Since scalar multiples of convergent series are convergent.). It follows that

\[
\sum_{n=1}^{\infty} (a_n + |a_n|) + (-|a_n|) = \sum_{n=1}^{\infty} a_n
\]

converges since the sum of convergent series is convergent. This completes the proof.