Math 21C
Kouba

Alternating Series Test (For Convergence Only)

**DEFINITION**: A series of the form

\[ \sum_{n=1}^{\infty} (-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots, \]

where \( a_n > 0 \) for all values of \( n = 1, 2, 3, 4, \cdots \), is called an alternating series. How can we test this series for convergence? We will need the following fact, which is given without proof.

**FACT A** : Assume that the sequence \( \{b_n\} \) satisfies the following two conditions:

1. \( b_1 < b_2 < b_3 < b_4 < \cdots \), i.e., \( b_n < b_{n+1} \) for \( n = 1, 2, 3, 4, \cdots \) (The sequence is strictly increasing.) and
2. \( b_n < C \), a fixed constant, for \( n = 1, 2, 3, 4, \cdots \) (The sequence is bounded.).

Then \( \lim_{n \to \infty} b_n = L \) for some finite number \( L \).

**Alternating Series Test**: Consider the series \( \sum_{n=1}^{\infty} (-1)^{n+1}a_n \). If the sequence \( \{a_n\} \) is positive (+), decreasing (↓), and \( \lim_{n \to \infty} a_n = 0 \), then the series \( \sum_{n=1}^{\infty} (-1)^{n+1}a_n \) converges.

**Proof**: Use the sequence of partial sums (even and odd separately). Since \( \{a_n\} \) is positive and decreasing, the following must be true:

- \( s_2 = a_1 + (-a_2) < a_1 \),
- \( s_4 = a_1 + (-a_2 + a_3 + (-a_4) < a_1 \),
- \( s_6 = a_1 + (-a_2 + a_3 + (-a_4 + a_5) < a_1 \),
- \( \cdots \)
- and
- \( s_{2n} = a_1 + (-a_2 + a_3 + (-a_4 + \cdots + (-a_{2n-2} + a_{2n-1} + (-a_{2n}) < a_1 \), \)
- also
- \( s_2 = (a_1 - a_2) \),
- \( s_4 = (a_1 - a_2) + (a_3 - a_4) = s_2 + (a_3 - a_4) > s_2 \),
- \( s_6 = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) = s_4 + (a_5 - a_6) > s_4 \), \cdots
and
\[ s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) = s_{2n-2} + (a_{2n-1} - a_{2n}) > s_{2n-2} , \cdots \]

thus, the sequence of even partial sums \( \{s_{2n}\} = \{s_2, s_4, s_6, s_8, \cdots\} \) is increasing and bounded above by \( a_1 \). It follows from FACT A that
\[ \lim_{n \to \infty} s_{2n} = L \], for some finite number \( L \).

Now consider the sequence of odd partial sums \( \{s_{2n-1}\} = \{s_1, s_3, s_5, s_7, \cdots\} \). Note that
\[
\begin{align*}
\begin{align*}
s_{2n} &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n} \\
&= (a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1}) - a_{2n} \\
&= s_{2n-1} - a_{2n} ,
\end{align*}
\end{align*}
\]
i.e.,
\[ s_{2n-1} = s_{2n} + a_{2n} . \]

Taking the limit of both sides we get
\[ \lim_{n \to \infty} s_{2n-1} = \lim_{n \to \infty} s_{2n} + \lim_{n \to \infty} a_{2n} = L + 0 = L . \]

Now the sequence of all partial sums \( \{s_n\} = \{s_1, s_2, s_3, s_4, s_5, \cdots\} \) satisfies \( \lim_{n \to \infty} s_n = L \).

Thus,
\[ \sum_{n=1}^{\infty} (-1)^{n+1} a_n = \lim_{n \to \infty} s_n = L \]

and the alternating series converges by the sequence of partial sums test.
Analysis of the Error (Remainder), $R_n$, for a Convergent Alternating Series

Begin by separating the $n$th partial sum, $s_n$, and the error (remainder), $R_n$, for a convergent alternating series:

\[
\sum_{n=1}^{\infty} (-1)^{n+1}a_n = a_1 - a_2 + a_3 - \cdots + (-1)^n a_n + (-1)^{n+1} a_{n+1} + (-1)^{n+2} a_{n+2} + \cdots
\]

partial sum, $s_n$  
error (remainder), $R_n$

It follows that the error satisfies

\[
R_n = \begin{cases} 
  a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots, & \text{if } n \text{ is even} \\
  -a_{n+1} + a_{n+2} - a_{n+3} + a_{n+4} - \cdots, & \text{if } n \text{ is odd} 
\end{cases}
\]

Using the same $(+)/(-)$ analysis as in the proof of the Alternating Series Test, we can conclude that ...

1.) If $n$ is even, then $0 < R_n < a_{n+1}$.

2.) If $n$ is odd, then $-a_{n+1} < R_n < 0$.

In general, it is then true that

\[-a_{n+1} < R_n < a_{n+1},\]

i.e.,

\[|R_n| < a_{n+1}\]

for $n = 1, 2, 3, 4, \cdots$. 