Partial Derivatives and Implicit Differentiation: Assume that function $F(x, y) = c$, where $c$ is a constant and $y = g(x)$, is an equation in $x$ and $y$. We will show here a new way to find the ordinary derivative $y' = \frac{dy}{dx}$ using the Chain Rule for partial derivatives. From the diagram and the Chain Rule we get

$$\frac{dF(x, y)}{dx} = \frac{dc}{dx} \quad \rightarrow$$

$$\frac{dF}{dx} = F_x \cdot \frac{dx}{dx} + F_y \cdot \frac{dy}{dx} = 0 \quad \rightarrow$$

$$F_x \cdot 1 + F_y \cdot \frac{dy}{dx} = 0 \quad \rightarrow \quad (\#) \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

**DEFINITION**: (I.) Let $z = f(x, y)$ be a function. The *gradient vector* of $f$ at the point $(x, y)$ is

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

(II.) Let $w = f(x, y, z)$ be a function. The *gradient vector* of $f$ at the point $(x, y, z)$ is

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

**THEOREM** (Gradient Vectors and Level Curves): (I.) Let $z = f(x, y)$ be a function and let $(x, y) = (a, b)$ with $f(a, b) = c$. Then the gradient vector $\nabla f(a, b)$ is normal (perpendicular) to the *level curve* $f(x, y) = c$. 

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**Proof**: The slope of the line tangent to the level curve at \((a, b)\) is

\[
\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x}
\]

so that a vector parallel to the tangent line at \((a, b)\) is

\[\vec{T} = (f_y, -f_x)^T .\]

(Recall that vectors \(\vec{v}\) and \(\vec{w}\) are orthogonal (perpendicular) iff \(\vec{v} \cdot \vec{w} = 0\).) Then

\[\vec{T} \cdot \nabla f(a, b) = (f_y, -f_x)^T \cdot (f_x^T, f_y^T) = f_x f_y - f_x f_y = 0 .\]

QED

**Theorem (Gradient Vectors and Level Curves)**: (II.) Let \(w = f(x, y, z)\) be a function and let \((x, y, z) = (a, b, c)\) with \(f(a, b, c) = d\). Then the gradient vector \(\nabla f(a, b, c)\) is normal (perpendicular) to the level curve \(f(x, y, z) = d\). 

**Proof**: This requires the concept of a vector function, which will be covered in Math 21D.